Effective Bosonic Degrees of Freedom for One-Flavour Chromodynamics

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Abstract

We apply an earlier formulated programme for quantization of nonabelian gauge theories to one-flavour chromodynamics. This programme consists in a complete reformulation of the functional integral in terms of gauge invariant quantities. For the model under consideration two types of gauge invariants occur – quantities, which are bilinear in quarks and antiquarks (mesons) and a matrix-valued covector field, which is bilinear in quarks, antiquarks and their covariant derivatives. This covector field is linear in the original gauge potential, and can be, therefore, considered as the gauge potential "dressed" in a gauge invariant way with matter. Thus, we get a complete bosonization of the theory. The strong interaction is described by a highly non-linear effective action obtained after integrating out quarks and gluons from the functional integral. All constructions are done consequently on the quantum level, where quarks and antiquarks are anticommuting objects. Our quantization procedure circumvents the Gribov ambiguity.

1 Introduction

This paper is a continuation of [1] and [2]. In [1] we have proved that the classical Dirac-Maxwell system can be reformulated in a spin-rotation covariant way in terms of gauge invariant quantities and in [2] we have shown that it is possible to perform similar

constructions on the level of the (formal) functional integral of Quantum Electrodynamics. As a result we obtain a functional integral completely reformulated in terms of local gauge invariant quantities, which differs essentially from the effective functional integral obtained via the Faddeev-Popov procedure [3]. In particular, it turns out that standard perturbation techniques, based upon a splitting of the effective Lagrangian into a free part (Gaussian measure) and an interaction part (proportional to the bare coupling constant e), are rather not applicable to this functional integral. On the contrary, our formulation seems to be rather well adapted to investigations of nonperturbative aspects of QED, for a first contribution of this type see [4].

In this paper we show that our programme can be also applied to a nonabelian gauge theory, namely Quantum Chromodynamics with one flavour. As in the case of QED, we end up with a description in terms of a set $(j^{ab}, {c_{\mu K}}^L)$ of purely bosonic invariants, where j^{ab} is built from bilinear combinations of quarks and antiquarks (mesons) and ${c_{\mu K}}^L$ is a set of complex-valued vector bosons built from the gauge potential and the quark fields. A naive counting of the degrees of freedom encoded in these quantities yields the correct result: The field j^{ab} is Hermitean and carries, therefore, 16 degrees of freedom, whereas ${c_{\mu K}}^L$ is complex-valued and carries, therefore, 32 degrees of freedom. On the other hand, the original configuration $\left({A_{\mu}}_A{}^B, \psi^a_A\right)$ carries $32 + 24 = 56$ degrees of freedom. Thus, exactly 8 gauge degrees of freedom have been removed. The main difficulty in our construction comes, of course, from the fact that the quark fields are Grassmann-algebra-valued. Ignoring this for a moment, one can give a heuristical idea, how the invariant vector bosons ${c_{\mu K}}^L$ arise: They may be considered as built from the gauge potential and the "phase" of the matter field, "gauged away" in a similar way as within the unitary gauge fixing procedure for theories of nonabelian Higgs type. (As a matter of fact, in our construction not only the "phase", but the whole matter field enters.) In reality, this simple-minded gauge fixing philosophy cannot be applied to Grassmann-algebra-valued objects. Instead of that one has to start (in some sense) with all invariants one can write down. Next one finds identities relating these invariants, which however — due to their Grassmann character — cannot be "solved" with respect to the correct number of effective invariants. But we show that there exists a scheme, which enables us to implement these identities under the functional integral and to integrate out the original quarks and gauge bosons. This is the main idea of the present paper. As a result we obtain a functional integral in terms of the correct number of effective gauge invariant bosonic quantities. Thus, our procedure consists in a certain reduction to a sector, where we have mesons j^{ab}, whose interaction is mediated by vector bosons ${c_{\mu K}}^L$.

We stress that our approach circumvents any gauge fixing procedure — and, therefore, also the Gribov problem, see [5] – [8]. The whole theory, including the pure Yang-Mills action is rewritten in terms of invariants. An important property of the effective theory we obtain is that it is highly non-linear. This is a consequence of integrating out quarks and gluons. Thus, as in the case of QED, it is doubtful whether perturbation techniques can be applied here. The natural next step will be rather to develop a lattice approximation

of QCD within this formulation.

Finally, we mention that our general programme of reformulating gauge theories in terms of invariants has been earlier applied to theories with bosonic matter fields (Higgs models), both for the continuum case [9], [10] and on the lattice [11].

The paper is organized as follows: In Section 2 we introduce basic notations and define gauge invariant quantities, built from the gauge potential and the (anticommuting) quark fields. Moreover, we prove some algebraic identities relating these invariants. In Section 3 we derive basic identities relating the Lagrangian with these invariants. Finally, in Section 4 we show how to implement the above mentioned identities under the functional integral to obtain an effective functional integral in terms of invariants. The paper is completed by two Appendices, where we give a review of spin tensor algebraic tools used in this paper, and present some technical points skipped in the text.

2 Basic Notations and Gauge Invariants

A field configuration of one-flavour chromodynamics consists of an SU(3)-gauge potential $\left({A_\mu}_A{}^B\right)$ and a four-component colored quark field (ψ^a_A), where $A, B, ..., J = 1, 2, 3$ are color indices, $a, b, ... = 1, 2, \dot{1}, \dot{2}$ denote bispinor indices and $\mu, \nu, ... = 0, 1, 2, 3$ spacetime indices.

Ordinary spinor indices are denoted by $K, L, ... = 1, 2$. The components of (ψ^a_A) are anticommuting (Grassmann-algebra valued) quantities and build up a Grassmann-algebra of (pointwise real) dimension 24.

The one-flavour chromodynamics Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_{mat}, \tag{2.1}$$

with

$$\begin{aligned} \mathcal{L}_{gauge} &= -\frac{1}{8} {F_{\mu\nu}}_A{}^B {F^{\mu\nu}}_B{}^A, & (2.2)\\ \mathcal{L}_{mat} &= -m\, \overline{\psi^a_A}\, \beta_{ab}\, g^{AB}\, \psi^b_B - \mathrm{Im}\left\{ \overline{\psi^a_A}\, \beta_{ab}\, (\gamma^\mu)^b{}_c\, g^{AB}\, (D_\mu \psi^c_B) \right\}, & (2.3) \end{aligned}$$

where

$$\begin{aligned} {F_{\mu\nu}}_A{}^B &= \partial_\mu {A_\nu}_A{}^B - \partial_\nu {A_\mu}_A{}^B + ig\, [A_\mu, A_\nu]_A{}^B, & (2.4)\\ D_\mu \psi^a_A &= \partial_\mu \psi^a_A + ig\, {A_\mu}_A{}^B\, \psi^a_B & (2.5) \end{aligned}$$

are the field strength and the covariant derivative. In contrast to standard notation, the bar denotes in this paper complex conjugation, g^{AB} and β_{ab} denote the Hermitian metrics in color and bispinor space respectively and $(\gamma^\mu)^b{}_c$ are the Dirac matrices (see Appendix A). The starting point for formulating the quantum theory is the formal functional measure

$$\mathcal{F} = \int \prod \mathrm{d}\psi \prod \mathrm{d}\overline{\psi} \prod \mathrm{d}A\, \mathrm{e}^{i\, S\left[A,\psi,\overline{\psi}\right]}, \tag{2.6}$$

where

$$S[A,\psi,\overline{\psi}\,] = \int \mathrm{d}^4x\, \mathcal{L}\left[A,\psi,\overline{\psi}\,\right] \tag{2.7}$$

denotes the physical action. Here, the integral over the anticommuting fields ψ and $\overline{\psi}$ is understood in the sense of Berezin ([13], see also [14] – [16]). To calculate the vacuum expectation value for some observable, i. e. a gauge invariant function $\mathcal{O}[A,\psi,\overline{\psi}]$, one has to integrate this observable with respect to the above measure. The main result of our paper consists in reformulating the measure $\mathcal{F}$ in terms of effective, gauge invariant, degrees of freedom of the theory (see discussion at the end of Section 4).

Let us define the following fundamental gauge invariant Grassmann-algebra valued quantities:

$$\mathcal{J}^{ab} := \overline{\psi^a_A}\, g^{AB}\, \psi^b_B, \tag{2.8}$$

$$C^{ab}_\mu := \overline{\psi^a_A}\, g^{AB}\,(D_\mu\psi^b_B) - (\,\overline{D_\mu\psi^a_A}\,)\, g^{AB}\, \psi^b_B. \tag{2.9}$$

Obviously, $\mathcal{J}^{ab}$ is a Hermitian field of even (commuting) type (of rank 2 in the Grassmann-algebra):

$$\overline{\mathcal{J}^{ab}} = \overline{\overline{\psi^a_A}\, g^{AB}\, \psi^b_B} = \overline{\psi^b_B}\,\overline{g^{AB}}\,\psi^a_A = \overline{\psi^b_B}\, g^{BA}\, \psi^a_A = \mathcal{J}^{ba}, \tag{2.10}$$

whereas the field C^{ab}_μ is anti-Hermitian:

$$\overline{C^{ab}_\mu} = -\,C^{ba}_\mu. \tag{2.11}$$

We denote

$$\mathcal{J}^2 := \mathcal{J}^{ab}\beta_{acbd}\mathcal{J}^{cd}, \tag{2.12}$$

which is a scalar field of rank 4. Here we introduced

$$\beta_{abcd} := \frac{1}{2}\,\overline{(\gamma_\mu)_{ab}}\,(\gamma^\mu)_{cd}.$$

Moreover, for shortness of notation we define the following auxiliary invariants

$$\begin{aligned} \mathcal{X}^2 &:= 4\,\beta_{bcef}\,\beta_{ad}\,\left\{\mathcal{J}^{ad}\mathcal{J}^{be}\mathcal{J}^{cf} + 2\,\mathcal{J}^{ae}\mathcal{J}^{bf}\mathcal{J}^{cd}\right\} \\ &\equiv 4\,\mathcal{J}^{ad}\mathcal{J}^{be}\mathcal{J}^{cf}\,\left\{\beta_{bcef}\,\beta_{ad} + 2\,\beta_{bcde}\,\beta_{af}\right\}, \end{aligned} \tag{2.13}$$

and

$$B^{ab}_\mu := \overline{\psi^a_A}\, g^{AB}\,(D_\mu\psi^b_B), \tag{2.14}$$

which obey the identities

$$B^{ab}_\mu - \overline{B^{ba}_\mu} = C^{ab}_\mu, \tag{2.15}$$

$$B^{ab}_\mu + \overline{B^{ba}_\mu} = \partial_\mu\mathcal{J}^{ab}\,, \tag{2.16}$$

and consequently

$$B_\mu^{ab} = \frac{1}{2}\left((\partial_\mu \mathcal{J}^{ab}) + C_\mu^{ab}\right), \tag{2.17}$$

$$\overline{B_\mu^{ba}} = \frac{1}{2}\left((\partial_\mu \mathcal{J}^{ab}) - C_\mu^{ab}\right). \tag{2.18}$$

Proposition 1 *The following identities hold*

$$\begin{aligned}
&\mathcal{X}^2 \left(C_\mu^{ab} + (\partial_\mu \mathcal{J}^{ab})\right) \\
&= -4 \left(C_\mu^{cb} + (\partial_\mu \mathcal{J}^{cb})\right) \beta_{cf} \left(\mathcal{J}^2 \mathcal{J}^{af} + 2\,\beta_{ghde}\,\mathcal{J}^{gd}\mathcal{J}^{hf}\mathcal{J}^{ae}\right) \qquad (2.19)\\
&\quad -8 \left(C_\mu^{gb} + (\partial_\mu \mathcal{J}^{gb})\right) \beta_{cf}\,\beta_{ghde} \Big(\mathcal{J}^{he}\mathcal{J}^{cf}\mathcal{J}^{ad} + \mathcal{J}^{cd}\mathcal{J}^{hf}\mathcal{J}^{ae} + \mathcal{J}^{hd}\mathcal{J}^{ce}\mathcal{J}^{af}\Big).
\end{aligned}$$

Proof. To prove these identities we make use of the following relations:

$$\begin{aligned}
\overline{\epsilon}^{ABC}\,\epsilon^{DEF} &= g^{AD}\,g^{BE}\,g^{CF} + g^{AE}\,g^{BF}\,g^{CD} + g^{AF}\,g^{BD}\,g^{CE} \\
&\quad - g^{AF}\,g^{BE}\,g^{CD} - g^{AE}\,g^{BD}\,g^{CF} - g^{AD}\,g^{BF}\,g^{CE}, \qquad (2.20)\\
\epsilon^{ABC}\,g^{EF} &= \epsilon^{ABF}\,g^{EC} + \epsilon^{BCF}\,g^{EA} + \epsilon^{CAF}\,g^{EB}. \qquad (2.21)
\end{aligned}$$

Using the symmetry properties of β_{ghde} (see Appendix A) we first calculate

$$\begin{aligned}
\mathcal{X}^2\,B_\mu^{ab} &= \mathcal{X}^2\,g^{AB}\,\overline{\psi_A^a}\,(D_\mu\psi_B^b) \\
&= 4\,\beta_{ghde}\,\beta_{cf}\,\left\{\mathcal{J}^{cf}\mathcal{J}^{gd}\mathcal{J}^{he} + 2\,\mathcal{J}^{cd}\mathcal{J}^{ge}\mathcal{J}^{hf}\right\}\,g^{AB}\,\overline{\psi_A^a}\,(D_\mu\psi_B^b) \\
&= 4\,\beta_{ghde}\,\beta_{cf}\,\left\{g^{CF}g^{GD}g^{HE} + 2\,g^{CD}g^{GE}g^{HF}\right\}\,g^{AB} \\
&\quad \times \overline{\psi_C^c}\,\psi_F^f\,\overline{\psi_G^g}\,\psi_D^d\,\overline{\psi_H^h}\,\psi_E^e\,\overline{\psi_A^a}\,(D_\mu\psi_B^b) \\
&= 2\,\beta_{ghde}\,\beta_{cf}\,\Big\{g^{CF}g^{GD}g^{HE} + g^{CD}g^{GE}g^{HF} + g^{CE}g^{GF}g^{HD} \\
&\quad - g^{CE}g^{GD}g^{HF} - g^{CF}g^{GE}g^{HD} - g^{CD}g^{GF}g^{HE}\Big\}\,g^{AB} \\
&\quad \times \overline{\psi_C^c}\,\psi_F^f\,\overline{\psi_G^g}\,\psi_D^d\,\overline{\psi_H^h}\,\psi_E^e\,\overline{\psi_A^a}\,(D_\mu\psi_B^b) \\
&= 2\,\beta_{ghde}\,\beta_{cf}\,\overline{\epsilon}^{CGH}\,\epsilon^{FDE}\,g^{AB}\,\overline{\psi_C^c}\,\psi_F^f\,\overline{\psi_G^g}\,\psi_D^d\,\overline{\psi_H^h}\,\psi_E^e\,\overline{\psi_A^a}\,(D_\mu\psi_B^b) \\
&= 2\,\beta_{ghde}\,\beta_{cf}\,\overline{\epsilon}^{CGH}\,\left\{\epsilon^{FDB}\,g^{AE} + \epsilon^{DEB}\,g^{AF} + \epsilon^{EFB}\,g^{AD}\right\} \\
&\quad \times \overline{\psi_C^c}\,\psi_F^f\,\overline{\psi_G^g}\,\psi_D^d\,\overline{\psi_H^h}\,\psi_E^e\,\overline{\psi_A^a}\,(D_\mu\psi_B^b) \\
&= 2\,\beta_{ghde}\,\beta_{cf}\,\left\{2\,\overline{\epsilon}^{CGH}\,\epsilon^{FDB}\,g^{AE} + \overline{\epsilon}^{CGH}\,\epsilon^{DEB}\,g^{AF}\right\} \\
&\quad \times \overline{\psi_C^c}\,\psi_F^f\,\overline{\psi_G^g}\,\psi_D^d\,\overline{\psi_H^h}\,\psi_E^e\,\overline{\psi_A^a}\,(D_\mu\psi_B^b) \\
&= 2\,\beta_{ghde}\,\beta_{cf}\,\Big\{2\,g^{CF}g^{GD}g^{HB}g^{AE} + 2\,g^{CD}g^{GB}g^{HF}g^{AE} + 2\,g^{CB}g^{GF}g^{HD}g^{AE} \\
&\quad - 2\,g^{CB}g^{GD}g^{HF}g^{AE} - 2\,g^{CD}g^{GF}g^{HB}g^{AE} - 2\,g^{CF}g^{GB}g^{HD}g^{AE}
\end{aligned}$$

$$
\begin{aligned}
& +g^{CD}g^{GE}g^{HB}g^{AF}+g^{CE}g^{GB}g^{HD}g^{AF}+g^{CB}g^{GD}g^{HE}g^{AF} \\
& -g^{CB}g^{GE}g^{HD}g^{AF}-g^{CE}g^{GD}g^{HB}g^{AF}-g^{CD}g^{GB}g^{HE}g^{AF}\Big\} \\
& \times\overline{\psi^c_C}\,\psi^f_F\,\overline{\psi^g_G}\,\psi^d_D\,\overline{\psi^h_H}\,\psi^e_E\,\overline{\psi^a_A}\,(D_\mu\psi^b_B) \\
= \; & -4\,\beta_{ghde}\,\beta_{cf}\,\{2\,\mathcal{J}^{cf}\mathcal{J}^{gd}\mathcal{J}^{ae}B^{hb}_\mu+2\,\mathcal{J}^{cd}\mathcal{J}^{hf}\mathcal{J}^{ae}B^{gb}_\mu \\
& +2\,\mathcal{J}^{gf}\mathcal{J}^{hd}\mathcal{J}^{ae}B^{cb}_\mu+\mathcal{J}^{cd}\mathcal{J}^{ge}\mathcal{J}^{af}B^{hb}_\mu \\
& +\mathcal{J}^{ce}\mathcal{J}^{hd}\mathcal{J}^{af}B^{gb}_\mu+\mathcal{J}^{gd}\mathcal{J}^{he}\mathcal{J}^{af}B^{cb}_\mu\} \\
= \; & -4\,B^{cb}_\mu\,\beta_{cf}\left(\mathcal{J}^2\mathcal{J}^{af}+2\,\beta_{ghde}\,\mathcal{J}^{gd}\mathcal{J}^{hf}\mathcal{J}^{ae}\right) \\
& -8\,B^{gb}_\mu\,\beta_{ghde}\,\beta_{cf}\left(\mathcal{J}^{he}\mathcal{J}^{cf}\mathcal{J}^{ad}+\mathcal{J}^{cd}\mathcal{J}^{hf}\mathcal{J}^{ae}+\mathcal{J}^{hd}\mathcal{J}^{ce}\mathcal{J}^{af}\right).
\end{aligned}
$$

Finally, inserting (2.17) we get (2.19). □

Using the block representation of $\mathcal{J}^{ab}$ and C^{ab}_μ (see Appendix A), equation (2.19) leads to four equations, written down in terms of spinor indices:

$$
\begin{aligned}
C_{\mu M}{}^{L}\,Q(\mathcal{J})^{\dot K M} &= C^{\dot M L}_\mu\left(\delta^{\dot K}{}_{\dot M}\,\mathcal{X}^2-Q(\mathcal{J})^{\dot K}{}_{\dot M}\right)-(\partial_\mu\mathcal{J}_M{}^L)\,Q(\mathcal{J})^{\dot K M} \\
&\quad -(\partial_\mu\mathcal{J}^{\dot M L})\,Q(\mathcal{J})^{\dot K}{}_{\dot M}+(\partial_\mu\mathcal{J}^{\dot K L})\,\mathcal{X}^2 && (2.22) \\
C_\mu{}^{\dot M}{}_{\dot L}\,Q(\mathcal{J})_{K\dot M} &= C_{\mu M\dot L}\left(\delta_K{}^M\,\mathcal{X}^2-Q(\mathcal{J})_K{}^M\right)-(\partial_\mu\mathcal{J}_{M\dot L})\,Q(\mathcal{J})_K{}^M \\
&\quad -(\partial_\mu\mathcal{J}^{\dot M}{}_{\dot L})\,Q(\mathcal{J})_{K\dot M}+(\partial_\mu\mathcal{J}_{K\dot L})\,\mathcal{X}^2 && (2.23) \\
C_\mu{}^{\dot M L}\,Q(\mathcal{J})_{K\dot M} &= C_{\mu M}{}^{L}\left(\delta_K{}^M\,\mathcal{X}^2-Q(\mathcal{J})_K{}^M\right)-(\partial_\mu\mathcal{J}_M{}^L)\,Q(\mathcal{J})_K{}^M \\
&\quad -(\partial_\mu\mathcal{J}^{\dot M L})\,Q(\mathcal{J})_{K\dot M}+(\partial_\mu\mathcal{J}_K{}^L)\,\mathcal{X}^2 && (2.24) \\
C_{\mu M\dot L}\,Q(\mathcal{J})^{\dot K M} &= C_\mu{}^{\dot M}{}_{\dot L}\left(\delta^{\dot K}{}_{\dot M}\,\mathcal{X}^2-Q(\mathcal{J})^{\dot K}{}_{\dot M}\right)-(\partial_\mu\mathcal{J}_{M\dot L})\,Q(\mathcal{J})^{\dot K M} \\
&\quad -(\partial_\mu\mathcal{J}^{\dot M}{}_{\dot L})\,Q(\mathcal{J})^{\dot K}{}_{\dot M}+(\partial_\mu\mathcal{J}^{\dot K}{}_{\dot L})\,\mathcal{X}^2 && (2.25) \\
& && (2.26)
\end{aligned}
$$

where

$$
\begin{aligned}
Q(\mathcal{J})_K{}^M &= 8\,(-\mathcal{J}^2\mathcal{J}_K{}^M+2\,\mathcal{J}_O{}^O\mathcal{J}^{\dot N M}\mathcal{J}_{K\dot N}+2\,\mathcal{J}^{\dot N}{}_{\dot N}\mathcal{J}_O{}^M\mathcal{J}_K{}^O \\
&\quad +2\,\mathcal{J}_O{}^M\mathcal{J}^{\dot N O}\mathcal{J}_{K\dot N}+2\,\mathcal{J}_{O\dot N}\mathcal{J}^{\dot N M}\mathcal{J}_K{}^O+2\,\mathcal{J}^{\dot N}{}_{\dot N}\mathcal{J}^{\dot O M}\mathcal{J}_{K\dot O} && (2.27) \\
&\quad +\mathcal{J}^{\dot N}{}_{\dot O}\mathcal{J}^{\dot O}{}_{\dot N}\mathcal{J}_K{}^M+\mathcal{J}^{\dot O}{}_{\dot O}\mathcal{J}^{\dot N}{}_{\dot N}\mathcal{J}_K{}^M+2\,\mathcal{J}^{\dot N M}\mathcal{J}^{\dot O}{}_{\dot N}\mathcal{J}_{K\dot O}) \\
Q(\mathcal{J})^{\dot K}{}_{\dot M} &= 8\,(-\mathcal{J}^2\mathcal{J}^{\dot K}{}_{\dot M}+2\,\mathcal{J}^{\dot N}{}_{\dot N}\mathcal{J}_{O\dot M}\mathcal{J}^{\dot K O}+2\,\mathcal{J}_O{}^O\mathcal{J}^{\dot N}{}_{\dot M}\mathcal{J}^{\dot K}{}_{\dot N} \\
&\quad +2\,\mathcal{J}^{\dot N}{}_{\dot M}\mathcal{J}_{O\dot N}\mathcal{J}^{\dot K O}+2\,\mathcal{J}_{O\dot M}\mathcal{J}^{\dot N O}\mathcal{J}^{\dot K}{}_{\dot N}+\mathcal{J}_N{}^N\mathcal{J}_O{}^O\mathcal{J}^{\dot K}{}_{\dot M} && (2.28) \\
&\quad +2\,\mathcal{J}_{N\dot M}\mathcal{J}_O{}^O\mathcal{J}^{\dot K N}+\mathcal{J}_O{}^N\mathcal{J}_N{}^O\mathcal{J}^{\dot K}{}_{\dot M}+2\,\mathcal{J}_{O\dot M}\mathcal{J}_N{}^O\mathcal{J}^{\dot K N}) \\
Q(\mathcal{J})^{\dot K M} &= 8\,(-\mathcal{J}^2\mathcal{J}^{\dot K M}+2\,\mathcal{J}_O{}^O\mathcal{J}^{\dot N M}\mathcal{J}^{\dot K}{}_{\dot N}+2\,\mathcal{J}^{\dot N}{}_{\dot N}\mathcal{J}_O{}^M\mathcal{J}^{\dot K O}
\end{aligned}
$$

$$
\begin{aligned}
& +2\,\mathcal{J}_O{}^M\mathcal{J}^{\dot N O}\mathcal{J}^{\dot K}{}_{\dot N}+2\,\mathcal{J}_{O\dot N}\mathcal{J}^{\dot N M}\mathcal{J}^{\dot K O}+2\,\mathcal{J}^{\dot N}{}_{\dot N}\mathcal{J}^{\dot O M}\mathcal{J}^{\dot K}{}_{\dot O} \qquad (2.29)\\
& +\mathcal{J}^{\dot N}{}_{\dot O}\mathcal{J}^{\dot O}{}_{\dot N}\mathcal{J}^{\dot K M}+\mathcal{J}^{\dot O}{}_{\dot O}\mathcal{J}^{\dot N}{}_{\dot N}\mathcal{J}^{\dot K M}+2\,\mathcal{J}^{\dot N M}\mathcal{J}^{\dot O}{}_{\dot N}\mathcal{J}^{\dot K}{}_{\dot O})
\end{aligned}
$$

$$
\begin{aligned}
Q(\mathcal{J})_{K\dot M} \;=\;& 8\,(-\mathcal{J}^2\mathcal{J}_{K\dot M}+2\,\mathcal{J}^{\dot N}{}_{\dot N}\mathcal{J}_{O\dot M}\mathcal{J}_K{}^O+2\,\mathcal{J}_O{}^O\mathcal{J}^{\dot N}{}_{\dot M}\mathcal{J}_{K\dot N}\\
& +2\,\mathcal{J}^{\dot N}{}_{\dot M}\mathcal{J}_{O\dot N}\mathcal{J}_K{}^O+2\,\mathcal{J}_{O\dot M}\mathcal{J}^{\dot N O}\mathcal{J}_{K\dot N}+\mathcal{J}_N{}^N\mathcal{J}_O{}^O\mathcal{J}_{K\dot M} \qquad (2.30)\\
& +2\,\mathcal{J}_{N\dot M}\mathcal{J}_O{}^O\mathcal{J}_K{}^N+\mathcal{J}_O{}^N\mathcal{J}_N{}^O\mathcal{J}_{K\dot M}+2\,\mathcal{J}_{O\dot M}\mathcal{J}_N{}^O\mathcal{J}_K{}^N).
\end{aligned}
$$

Later on, two of these equations will be used to eliminate half of the C_μ^{ab}-fields under the functional integral. One can show by a straightforward calculation that all components of the 2×2-matrices $Q(\mathcal{J})_K{}^M$, $Q(\mathcal{J})^{\dot K}{}_{\dot M}$, $Q(\mathcal{J})^{\dot K M}$ and $Q(\mathcal{J})_{K\dot M}$ do not vanish identically.

Finally, we note that

$$
C_\mu^{ab}=2ig\,\overline{\psi_A^a}\,g^{AB}\psi_C^b\,A_{\mu B}{}^C+\left(\overline{\psi_A^a}\,g^{AB}\,(\partial_\mu\psi_B^b)+\psi_B^b\,g^{AB}\,(\partial_\mu\overline{\psi_A^a})\right), \qquad (2.31)
$$

which can be seen by inserting the covariant derivative (2.5) into the definition of C_μ^{ab}.

3 The Lagrangian in Terms of Gauge Invariants

To reformulate the Lagrangian (2.1) in terms of the gauge invariants introduced above, we use the same ideas as in the case of QED (see [2]). In particular, for the calculation of $\mathcal{L}_{gauge}$ we have to find a nonvanishing element of maximal rank in the underlying Grassmann-algebra.

Lemma 1 *The quantity $(\mathcal{X}^2)^4$ is a nonvanishing element of maximal rank in the Grassmann-algebra.*

The proof of this Lemma is technical and can be found in Appendix B.

Next, let us introduce the following auxiliary variables convenient for further calculations:

$$
V^{Cab}:=\epsilon^{ABC}\,\psi_A^a\,\psi_B^b. \qquad (3.1)
$$

Moreover, we denote $\psi^{Aa}\equiv g^{AB}\,\psi_B^a$.

Lemma 2 *The following identity holds*

$$
\mathcal{X}^2\,g^{AB}=\overline{(\gamma_\mu)_{gh}}\,\overline{\epsilon}^{CGH}\,\overline{\psi_H^h}\,\overline{\psi_G^g}\,\overline{\psi_C^c}\,\psi^{Ad}\,V^{Bef}\,\{2\,\beta_{ce}\,(\gamma^\mu)_{fd}+\beta_{cd}\,(\gamma^\mu)_{fe}\}. \qquad (3.2)
$$

Proof. To prove (3.2), we make use of (2.21) as well as $V^{Cab}=V^{Cba}$ and the symmetry of β_{ghde}:

$$
\begin{aligned}
\mathcal{X}^2\,g^{AB} &= 4\,\beta_{ghde}\,\beta_{cf}\,\left\{\mathcal{J}^{cf}\mathcal{J}^{gd}\mathcal{J}^{he}+2\,\mathcal{J}^{cd}\mathcal{J}^{ge}\mathcal{J}^{hf}\right\}\,g^{AB}\\
&= 4\,\beta_{ghde}\,\beta_{cf}\,\left\{g^{CF}g^{GD}g^{HE}+2\,g^{CD}g^{GE}g^{HF}\right\}\,g^{AB}\,\overline{\psi_C^c}\,\psi_F^f\,\overline{\psi_G^g}\,\psi_D^d\,\overline{\psi_H^h}\,\psi_E^e
\end{aligned}
$$

$$
\begin{aligned}
&= 2\,\beta_{ghde}\,\beta_{cf}\,\Big\{g^{CF}g^{GD}g^{HE}+g^{CD}g^{GE}g^{HF}+g^{CE}g^{GF}g^{HD}-g^{CE}g^{GD}g^{HF}\\
&\quad -g^{CF}g^{GE}g^{HD}-g^{CD}g^{GF}g^{HE}\Big\}\,g^{AB}\,\overline{\psi^c_C}\,\psi^f_F\,\overline{\psi^g_G}\,\psi^d_D\,\overline{\psi^h_H}\,\psi^e_E\\
&= 2\,\beta_{ghde}\,\beta_{cf}\,\overline{\epsilon}^{CGH}\,\epsilon^{FDE}\,g^{AB}\,\overline{\psi^c_C}\,\psi^f_F\,\overline{\psi^g_G}\,\psi^d_D\,\overline{\psi^h_H}\,\psi^e_E\\
&= 2\,\beta_{ghde}\,\beta_{cf}\,\overline{\epsilon}^{CGH}\,\left\{\epsilon^{FDB}\,g^{AE}+\epsilon^{DEB}\,g^{AF}+\epsilon^{EFB}\,g^{AD}\right\}\,\overline{\psi^c_C}\,\psi^f_F\,\overline{\psi^g_G}\,\psi^d_D\,\overline{\psi^h_H}\,\psi^e_E\\
&= \overline{(\gamma_\mu)_{gh}}\,(\gamma^\mu)_{de}\,\beta_{cf}\,\overline{\epsilon}^{CGH}\,\left\{2\,\epsilon^{FDB}\,g^{AE}+\epsilon^{DEB}\,g^{AF}\right\}\,\overline{\psi^c_C}\,\psi^f_F\,\overline{\psi^g_G}\,\psi^d_D\,\overline{\psi^h_H}\,\psi^e_E\\
&= \overline{(\gamma_\mu)_{gh}}\,\overline{\epsilon}^{CGH}\,\overline{\psi^h_H}\,\overline{\psi^g_G}\,\overline{\psi^c_C}\,\beta_{cf}\,(\gamma^\mu)_{de}\,\Big\{2\,V^{Bfd}\,\psi^e_E\,g^{AE}+V^{Bde}\,\psi^f_F\,g^{AF}\Big\}\\
&= \overline{(\gamma_\mu)_{gh}}\,\overline{\epsilon}^{CGH}\,\overline{\psi^h_H}\,\overline{\psi^g_G}\,\overline{\psi^c_C}\,\psi^{Ad}\,V^{Bef}\,\left\{2\,\beta_{ce}\,(\gamma^\mu)_{fd}+\beta_{cd}\,(\gamma^\mu)_{fe}\right\}.
\end{aligned}
$$

□

Proposition 2 *We have*

$$(\mathcal{X}^2)^4\,\mathcal{L}_{gauge}=\frac{1}{8}\,(G_{\mu\nu})_{ab}\,(G^{\mu\nu})_{cd}\,\epsilon^{bc}\,\epsilon^{da} \tag{3.3}$$

with

$$
\begin{aligned}
(G_{\mu\nu})_{ab} \;:=\; & \frac{2\,\mathcal{X}^2}{ig}\,\epsilon_{ha}\,\beta_{cdefgb}\,\mathcal{J}^{cf}\mathcal{J}^{dg}(\partial_{[\mu}C^{eh}_{\nu]})\\
&+\frac{4}{ig}\,\epsilon_{ha}\,\beta_{cdefgb}\,\beta_{klmnop}\,\mathcal{J}^{cf}\mathcal{J}^{dg}\mathcal{J}^{kn}\mathcal{J}^{lo}\\
&\quad\times\{(\partial_{[\mu}\mathcal{J}^{ep})(\partial_{\nu]}\mathcal{J}^{mh})-C^{ep}_{[\mu}\,(\partial_{\nu]}\mathcal{J}^{mh})+(\partial_{[\mu}\mathcal{J}^{ep})\,C^{mh}_{\nu]}-C^{ep}_{[\mu}\,C^{mh}_{\nu]}\},
\end{aligned} \tag{3.4}
$$

where

$$\beta_{abcdef}:=2\,\{\beta_{abde}\beta_{cf}+2\,\beta_{abef}\beta_{cd}+\beta_{bcde}\beta_{af}+2\,\beta_{bcef}\beta_{ad}+\beta_{cade}\beta_{bf}+2\,\beta_{caef}\beta_{bd}\}. \tag{3.5}$$

Moreover, the matter Lagrangian $\mathcal{L}_{mat}$ takes the form

$$\mathcal{L}_{mat}=-m\,\beta_{ab}\,\mathcal{J}^{ab}-\frac{1}{2}\,\mathrm{Im}\left\{\beta_{ab}\,(\gamma^\mu)^b{}_c\,(\partial_\mu\mathcal{J}^{ac})+\beta_{ab}\,(\gamma^\mu)^b{}_c\,C^{ac}_\mu\right\}. \tag{3.6}$$

Proof. To prove this Proposition we make use of identity (3.2):

$$
\begin{aligned}
&(\mathcal{X}^2)^4\,\mathcal{L}_{gauge}\\
&= -\frac{1}{8}\,(\mathcal{X}^2)^4\,F_{\mu\nu A}{}^{B}F^{\mu\nu}{}_{B}{}^{A},\\
&= -\frac{1}{8}\,(\mathcal{X}^2)^2\,\left(F_{\mu\nu A}{}^{B}\mathcal{X}^2\right)\left(F^{\mu\nu}{}_{B}{}^{A}\mathcal{X}^2\right)\\
&= -\frac{1}{8}\,(\mathcal{X}^2)^2\,\left(F_{\mu\nu AC}\,g^{CB}\,\mathcal{X}^2\right)\left(F^{\mu\nu}{}_{BD}\,g^{DA}\,\mathcal{X}^2\right)\\
&= -\frac{1}{8}\,(\mathcal{X}^2)^2\,\Big(F_{\mu\nu AC}\,\overline{(\gamma_\sigma)_{fg}}\,\overline{\epsilon}^{EFG}\,\overline{\psi^g_G}\,\overline{\psi^f_F}\,\overline{\psi^e_E}\,\psi^C_m\,\epsilon^{cm}\,V^{Bab}\,\{2\,\beta_{ea}\,(\gamma^\sigma)_{bc}+\beta_{ec}\,(\gamma^\sigma)_{ba}\}\Big)\\
&\quad\times\Big(F^{\mu\nu}_{BD}\,\overline{(\gamma_\delta)_{ij}}\,\overline{\epsilon}^{HIJ}\,\overline{\psi^j_J}\,\overline{\psi^i_I}\,\overline{\psi^h_H}\,\psi^D_n\,\epsilon^{dn}\,V^{Akl}\,\{2\,\beta_{hk}\,(\gamma^\delta)_{ld}+\beta_{hd}\,(\gamma^\delta)_{lk}\}\Big).
\end{aligned}
$$

A reordering of factors yields

$$
\begin{aligned}
(\mathcal{X}^2)^4\,\mathcal{L}_{gauge} &= \frac{1}{8}\,\Big(\mathcal{X}^2\,V^{Akl}\,\overline{(\gamma_\delta)_{ij}}\,\overline{\epsilon}^{HIJ}\,\overline{\psi^j_J}\,\overline{\psi^i_I}\,\overline{\psi^h_H}\,F_{\mu\nu AC}\,\psi^C_m\,\gamma^\delta_{hkld}\Big)\,\epsilon^{cm} \\
&\quad\times\Big(\mathcal{X}^2\,V^{Bab}\,\overline{(\gamma_\sigma)_{fg}}\,\overline{\epsilon}^{EFG}\,\overline{\psi^g_G}\,\overline{\psi^f_F}\,\overline{\psi^e_E}\,F^{\mu\nu}{}_{BD}\,\psi^D_n\,\gamma^\sigma_{eabc}\Big)\,\epsilon^{dn} \\
&=: \frac{1}{8}\,(G_{\mu\nu})_{md}\,\epsilon^{cm}\,(G^{\mu\nu})_{nc}\,\epsilon^{dn}, \qquad (3.7)
\end{aligned}
$$

where we introduced

$$\gamma^\mu_{abcd} := 2\,\beta_{ab}\,(\gamma^\mu)_{cd} + \beta_{ad}\,(\gamma^\mu)_{cb}.$$

It remains to calculate $(G_{\mu\nu})_{md}$. Using

$$F_{\mu\nu AC}\psi^{Cb} \;\equiv\; \frac{1}{ig}\,(D_{[\mu}D_{\nu]}\psi^b_A),$$

we obtain

$$
\begin{aligned}
(G_{\mu\nu})_{md} &= \mathcal{X}^2\,V^{Akl}\,\overline{(\gamma_\delta)_{ij}}\,\overline{\epsilon}^{HIJ}\,\overline{\psi^j_J}\,\overline{\psi^i_I}\,\overline{\psi^h_H}\,F_{\mu\nu AC}\,\psi^C_m\,\gamma^\delta_{hkld} \\
&= \frac{1}{ig}\,\mathcal{X}^2\,V^{Akl}\,\overline{(\gamma_\delta)_{ij}}\,\overline{\epsilon}^{HIJ}\,\overline{\psi^j_J}\,\overline{\psi^i_I}\,\overline{\psi^h_H}\,(D_{[\mu}D_{\nu]}\psi^a_A)\,(\gamma^\delta_{hkld})\,\epsilon_{am} \\
&= \frac{2}{ig}\,\mathcal{X}^2\,\gamma^\delta_{hkld}\,\epsilon_{am}\,\overline{(\gamma_\delta)_{ij}}\,\psi^k_K\,\psi^l_L\,\overline{\psi^j_J}\,\overline{\psi^i_I}\,\overline{\psi^h_H}\,\big(D_{[\mu}D_{\nu]}\psi^a_A\big) \\
&\quad\times\{g^{HK}g^{IL}g^{JA} + g^{HL}g^{IA}g^{JK} + g^{HA}g^{IK}g^{JL}\}.
\end{aligned}
$$

In the last step the definition of V^{Cab} and the identity (2.20) were inserted. Moreover, the antisymmetry in the color indices (I, J) and the symmetry in the bispinor indices (i, j) were used. Changing the indices in each term of the above sum separately, the last equation gives

$$
\begin{aligned}
(G_{\mu\nu})_{md} &= \frac{2}{ig}\,\mathcal{X}^2\,g^{HK}g^{IL}g^{JA}\,\psi^k_K\,\psi^l_L\,\overline{\psi^j_J}\,\overline{\psi^i_I}\,\overline{\psi^h_H}\,\big(D_{[\mu}D_{\nu]}\psi^a_A\big) \\
&\quad\times\{\gamma^\delta_{hkld}\,\overline{(\gamma_\delta)_{ij}} + \gamma^\delta_{ikld}\,\overline{(\gamma_\delta)_{jh}} + \gamma^\delta_{jkld}\,\overline{(\gamma_\delta)_{hi}}\,\}\,\epsilon_{am} \\
&= \frac{2}{ig}\,\mathcal{X}^2\,\mathcal{J}^{hk}\mathcal{J}^{il}\,\overline{\psi^j_J}\,g^{JA}\,\big(D_{[\mu}D_{\nu]}\psi^a_A\big)\,\beta_{hijkld}\,\epsilon_{am},
\end{aligned}
$$

where β_{hijkld} is given by equation (3.5). With (2.14) and (3.2) we get

$$
\begin{aligned}
(G_{\mu\nu})_{md} &= \frac{2}{ig}\,\mathcal{X}^2\,\mathcal{J}^{hk}\mathcal{J}^{il}\Big\{D_{[\mu}\Big(\overline{\psi^j_J}\,g^{JA}(D_{\nu]}\psi^a_A)\Big) - (D_{[\mu}\overline{\psi^j_J})\,g^{JA}(D_{\nu]}\psi^a_A)\Big\}\,\beta_{hijkld}\,\epsilon_{am} \\
&= \frac{2}{ig}\,\mathcal{X}^2\,\mathcal{J}^{hk}\mathcal{J}^{il}(D_{[\mu}B^{ia}_{\nu]})\,\beta_{hijkld}\,\epsilon_{am} \\
&\quad -\frac{2}{ig}\,\mathcal{X}^2\,g^{JA}\,\mathcal{J}^{hk}\mathcal{J}^{il}(D_{[\mu}\overline{\psi^j_J})\,(D_{\nu]}\psi^a_A)\,\beta_{hijkld}\,\epsilon_{am}
\end{aligned}
$$

$$
\begin{aligned}
= \; & \frac{2}{ig}\,\mathcal{X}^2\,\mathcal{J}^{hk}\mathcal{J}^{il}(D_{[\mu}B^{ia}_{\nu]})\,\beta_{hijkld}\,\epsilon_{am} \\
& -\frac{2}{ig}\,\overline{(\gamma_\sigma)_{fg}}\,\overline{\epsilon}^{EFG}\,\overline{\psi^g_G}\,\overline{\psi^f_F}\,\overline{\psi^e_E}\,\psi^{Jn}\,V^{Aop}\,\{2\,\beta_{eo}\,(\gamma^\mu)_{pn}+\beta_{en}\,(\gamma^\sigma)_{po}\} \\
& \qquad \times\mathcal{J}^{hk}\mathcal{J}^{il}(D_{[\mu}\overline{\psi^j_J})\,(D_{\nu]}\psi^a_A)\,\beta_{hijkld}\,\epsilon_{am} \\
= \; & \frac{2}{ig}\,\mathcal{X}^2\,\mathcal{J}^{hk}\mathcal{J}^{il}(D_{[\mu}B^{ia}_{\nu]})\,\beta_{hijkld}\,\epsilon_{am} \\
& +\frac{2}{ig}\,\mathcal{J}^{hk}\mathcal{J}^{il}\,\overline{\epsilon}^{EFG}\,\epsilon^{BCA}\,\overline{\psi^g_G}\,\overline{\psi^f_F}\,\overline{\psi^e_E}\,\psi^b_B\,\psi^c_C\,\overline{B^{nj}_{[\mu}}\,(D_{\nu]}\psi^a_A)\,\overline{(\gamma_\sigma)_{fg}}\,\beta_{hijkld}\,\epsilon_{am}\,\gamma^\sigma_{eopn} \\
= \; & \frac{2}{ig}\,\mathcal{X}^2\,\mathcal{J}^{hk}\mathcal{J}^{il}(D_{[\mu}B^{ia}_{\nu]})\,\beta_{hijkld}\,\epsilon_{am} \\
& +\frac{4}{ig}\,\mathcal{J}^{hk}\mathcal{J}^{il}\,g^{EB}g^{FC}g^{GA}\,\overline{\psi^g_G}\,\overline{\psi^f_F}\,\overline{\psi^e_E}\,\psi^b_B\,\psi^c_C\,\overline{B^{nj}_{[\mu}}\,(D_{\nu]}\psi^a_A)\,\beta_{hijkld}\,\beta_{efgbcn}\,\epsilon_{am},
\end{aligned}
$$

and, finally,

$$
\begin{aligned}
(G_{\mu\nu})_{md} \; = \; & \frac{2}{ig}\,\mathcal{X}^2\,\mathcal{J}^{hk}\mathcal{J}^{il}(\partial_{[\mu}B^{ia}_{\nu]})\,\beta_{hijkld}\,\epsilon_{am} \\
& +\frac{4}{ig}\,\mathcal{J}^{hk}\mathcal{J}^{il}\mathcal{J}^{eb}\mathcal{J}^{fc}\,\overline{B^{nj}_{[\mu}}\,B^{ga}_{\nu]}\,\beta_{hijkld}\,\beta_{efgbcn}\,\epsilon_{am}.
\end{aligned} \tag{3.8}
$$

The last step consists in the elimination of the auxiliary variables B^{ab}_μ. Inserting (2.17) and (2.18) into (3.8) and renaming some indices we get (3.4). Together with (3.7) this completes the proof of (3.3).

To show (3.6) we simply insert the definition of $\mathcal{J}^{ab}$ and B^{ab}_μ into $\mathcal{L}_{mat}$:

$$
\begin{aligned}
\mathcal{L}_{mat} \; &= \; -m\,\overline{\psi^a_A}\,\beta_{ab}\,g^{AB}\,\psi^b_B-\mathrm{Im}\left\{\overline{\psi^a_A}\,\beta_{ab}\,(\gamma^\mu)^b{}_c\,g^{AB}\,(D_\mu\psi^c_B)\right\} \\
&= \; -m\,\beta_{ab}\,\mathcal{J}^{ab}-\mathrm{Im}\{\beta_{ab}\,(\gamma^\mu)^b{}_c\,B^{ac}_\mu\}
\end{aligned}
$$

Using again (2.17) to eliminate B^{ab}_μ, we obtain equation (3.6). □

Remarks:

1. Formula (3.3) is an identity on the level of elements of maximal rank in the underlying Grassmann-algebra. Since the space of elements of maximal rank is one-dimensional, dividing by a non-vanishing element of maximal rank is a well-defined operation giving a c-number. Thus, knowing $(\mathcal{X}^2)^4\,\mathcal{L}_{gauge}$, we can reconstruct $\mathcal{L}_{gauge}$ uniquely just dividing by $(\mathcal{X}^2)^4$. This means that Proposition 2 gives us the Lagrangian in terms of invariants $\mathcal{J}^{ab}$ and C^{ab}_μ.

2. The additional algebraic identities (2.19), which are basic for getting the correct number of degrees of freedom, cannot be "solved" on the level of the algebra of

Grassmann-algebra-valued invariants. However, as will be shown in the next section, it is possible to implement them under the functional integral. This enables us to eliminate half the number of components of C_μ^{ab}. The result will be an effective functional integral in terms of the correct number of degrees of freedom, see also the Introduction for a discussion of this point.

4 The Functional Integral

Now we start to reformulate the functional integral (2.6). For that purpose we will use the following notion of the δ-distribution on superspace

$$\delta(u-U) = \int e^{2\pi i\,\xi(u-U)}\,\mathrm{d}\xi = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\,\delta^{(n)}(u)U^n, \tag{4.1}$$

where u is a c-number variable and U a combination of Grassmann variables ψ and $\overline{\psi}$ with rank smaller or equal to the maximal rank. Due to the nilpotent character of U the above sum is finite. This δ-distribution is a special example of a vector-space-valued distribution in the sense of [17]. From the above definition we have immediately

$$1 = \int \delta(u-U)\,\mathrm{d}u. \tag{4.2}$$

One easily shows the following

Lemma 3 *For an arbitrary smooth function f we have*

$$f(u)\,\delta(u-U) = f(U)\,\delta(u-U). \tag{4.3}$$

The above observation leads to a technique, which frequently will be used in this section:

$$\begin{aligned}\int f(\alpha)\,\delta(u-U)\,\mathrm{d}u &= \int f\left(\alpha\,\frac{g(u)}{g(\tilde u)}\right)\,\delta(u-U)\,\delta(u-\tilde u)\,\mathrm{d}u\,\mathrm{d}\tilde u \\ &= \int f\left(\frac{\alpha\,g(U)}{g(u)}\right)\,\delta(u-U)\,\delta(U-\tilde u)\,\mathrm{d}u\,\mathrm{d}\tilde u.\end{aligned} \tag{4.4}$$

Here, α denotes any c-number or Grassmann-algebra valued quantity and $g(u)$ is an arbitrary smooth function of u, such that the rank of $\alpha\,g(U)$ is smaller or equal to the maximal rank of the underlying Grassmann-algebra.

Now we introduce new, independent fields (j,c) associated with the invariants $(\mathcal{J}^{ab}$, $C_\mu^{ab})$ in a sense, which will become clear in what follows. Both fields j and c are by definition bosonic and gauge-invariant; $j = (j^{ab})$ is a (c-number-valued) Hermitean spin tensor field of second rank and $c = (c_\mu^{ab})$ is a (c-number-valued) anti-Hermitean spin-tensor-valued covector field. We call (j,c) c-number mates of (J,C).

As mentioned earlier, equations (2.22) – (2.25) can be used to substitute half the number of components of the covector field C_μ^{ab}. We choose $C_{\mu K}{}^L$ as independent variables. Thus (2.24) and (2.25) can be used to eliminate all components in the diagonal blocks of C_μ^{ab} (see Appendix A). In the subspace of our bispinorspace, which corresponds to the elements of the off-diagonal blocks of the field C_μ^{ab}, we choose the following basis elements e_ρ, $\rho = 1 \ldots 8$:

$$
\begin{aligned}
&e_1 = \begin{pmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \quad
e_2 = \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \quad
e_3 = \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}\\
&e_4 = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix} \quad
e_5 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix} \quad
e_6 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0 \end{pmatrix}\\
&e_7 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{pmatrix} \quad
e_8 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0 \end{pmatrix}.
\end{aligned}
\tag{4.5}
$$

A basis in colorspace is given by the Gell–Mann–matrices t_α, where $\alpha = 1 \ldots 8$:

$$
\begin{aligned}
&t_1 = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix} \quad
t_2 = \begin{pmatrix} 0&-i&0\\i&0&0\\0&0&0 \end{pmatrix} \quad
t_3 = \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{pmatrix}\\
&t_4 = \begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix} \quad
t_5 = \begin{pmatrix} 0&0&-i\\0&0&0\\i&0&0 \end{pmatrix} \quad
t_6 = \begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix}\\
&t_7 = \begin{pmatrix} 0&0&0\\0&0&-i\\0&i&0 \end{pmatrix} \quad
t_8 = \tfrac{1}{\sqrt{3}} \begin{pmatrix} 1&0&0\\0&1&0\\0&0&-2 \end{pmatrix}.
\end{aligned}
\tag{4.6}
$$

Moreover, we denote

$$
\chi^2 = 4\, j^{ad} j^{be} j^{cf} \, \{\beta_{bcef}\, \beta_{ad} + 2\, \beta_{bcde}\, \beta_{af}\}\,, \quad , \tag{4.7}
$$

$$
j^2 = j^{ab}\, \beta_{acbd}\, j^{cd}, \tag{4.8}
$$

$$
M(j)_\rho{}^\sigma = 4g^2 \left\{ \frac{2}{3}\, j^a{}_b\, j^c{}_d + 2\, j^a{}_d\, j^c{}_b \right\} (e_\rho)_a{}^b\, (e^\sigma)_c{}^d, \tag{4.9}
$$

$$
Q(j)_K{}^M \;=\; 8\Big(- j^2 j_K{}^M + 2\, j_O{}^O j^{\dot N M} j_{K\dot N} + 2\, j^{\dot N}{}_{\dot N} j_O{}^M j_K{}^O
$$

$$\begin{aligned}
& +2\, j_O{}^M j^{\dot N O} j_{K\dot N} + 2\, j_{O\dot N} j^{\dot N M} j_K{}^O + 2\, j^{\dot N}{}_{\dot N} j^{\dot O M} j_{K\dot O} && (4.10)\\
& +j^{\dot N}{}_{\dot O} j^{\dot O}{}_{\dot N} j_K{}^M + j^{\dot O}{}_{\dot O} j^{\dot N}{}_{\dot N} j_K{}^M + 2\, j^{\dot N M} j^{\dot O}{}_{\dot N} j_{K\dot O}\Big),
\end{aligned}$$

$$\begin{aligned}
Q(j)^{\dot K}{}_{\dot M} \;=\;& 8\Big(- j^2 j^{\dot K}{}_{\dot M} + 2\, j^{\dot N}{}_{\dot N} j_{O\dot M} j^{\dot K O} + 2\, j_O{}^O j^{\dot N}{}_{\dot M} j^{\dot K}{}_{\dot N}\\
& +2\, j^{\dot N}{}_{\dot M} j_{O\dot N} j^{\dot K O} + 2\, j_{O\dot M} j^{\dot N O} j^{\dot K}{}_{\dot N} + j_N{}^N j_O{}^O j^{\dot K}{}_{\dot M} && (4.11)\\
& +2\, j_{N\dot M} j_O{}^O j^{\dot K N} + j_O{}^N j_N{}^O j^{\dot K}{}_{\dot M} + 2\, j_{O\dot M} j_N{}^O j^{\dot K N}\Big),
\end{aligned}$$

$$\begin{aligned}
Q(j)^{\dot K M} \;=\;& 8\Big(- j^2 j^{\dot K M} + 2\, j_O{}^O j^{\dot N M} j^{\dot K}{}_{\dot N} + 2\, j^{\dot N}{}_{\dot N} j_O{}^M j^{\dot K O}\\
& +2\, j_O{}^M j^{\dot N O} j^{\dot K}{}_{\dot N} + 2\, j_{O\dot N} j^{\dot N M} j^{\dot K O} + 2\, j^{\dot N}{}_{\dot N} j^{\dot O M} j^{\dot K}{}_{\dot O} && (4.12)\\
& +j^{\dot N}{}_{\dot O} j^{\dot O}{}_{\dot N} j^{\dot K M} + j^{\dot O}{}_{\dot O} j^{\dot N}{}_{\dot N} j^{\dot K M} + 2\, j^{\dot N M} j^{\dot O}{}_{\dot N} j^{\dot K}{}_{\dot O}\Big),
\end{aligned}$$

$$\begin{aligned}
Q(j)_{K\dot M} \;=\;& 8\Big(- j^2 j_{K\dot M} + 2\, j^{\dot N}{}_{\dot N} j_{O\dot M} j_K{}^O + 2\, j_O{}^O j^{\dot N}{}_{\dot M} j_{K\dot N}\\
& +2\, j^{\dot N}{}_{\dot M} j_{O\dot N} j_K{}^O + 2\, j_{O\dot M} j^{\dot N O} j_{K\dot N} + j_N{}^N j_O{}^O j_{K\dot M} && (4.13)\\
& +2\, j_{N\dot M} j_O{}^O j_K{}^N + j_O{}^N j_N{}^O j_{K\dot M} + 2\, j_{O\dot M} j_N{}^O j_K{}^N\Big).
\end{aligned}$$

Proposition 3 *The functional integral $\mathcal{F}$ in terms of the gauge invariant set (j, c) is given by*

$$\mathcal{F} = \int \prod \mathrm{d}j\, \mathrm{d}c\, K[j]\, \mathrm{e}^{iS[j,c]}, \tag{4.14}$$

with the integral kernel

$$\begin{aligned}
K[j] \;=\;& \frac{6^4}{12!}\, \frac{(\chi^2)^4}{\det[M(j)_\rho{}^\sigma]\, (j^2)^2}\, \epsilon_{c_1c_4c_7c_{10}}\, \epsilon_{c_2c_5c_8c_{11}}\, \epsilon_{c_3c_6c_9c_{12}}\, \epsilon^{b_1b_4b_7b_{10}}\, \epsilon^{b_2b_5b_8b_{11}}\, \epsilon^{b_3b_6b_9b_{12}}\\
& \times \left\{ \prod_{r=1}^{12} \beta^{a_rc_r}\, \frac{\partial}{\partial j^{a_rb_r}} \right\} \delta(j) && (4.15)
\end{aligned}$$

and

$$S[j, c] = \int \mathrm{d}^4x\, \mathcal{L}[j, c]\ , \tag{4.16}$$

with the effective Lagrangian $\mathcal{L}[j, c]$ given by

$$\begin{aligned}
\mathcal{L}[j, c] \;=\;& \frac{1}{8\,(\chi^2)^4}\, (G_{\mu\nu})_{ab}\, (G^{\mu\nu})_{cd}\, \epsilon^{bc}\, \epsilon^{da} - m\, \beta_{ab}\, j^{ab}\\
& -\frac{1}{2}\, \mathrm{Im}\left\{ \beta_{ab}\, (\gamma^\mu)^b{}_c\, (\partial_\mu j^{ac}) + \beta_{ab}\, (\gamma^\mu)^b{}_c\, c^{ac}_\mu \right\}\ , && (4.17)
\end{aligned}$$

where

$$
\begin{aligned}
(G_{\mu\nu})_{ab} &= \frac{2\,\chi^2}{ig}\,\epsilon_{ha}\,\beta_{cdefgb}\,j^{cf}\,j^{dg}(\partial_{[\mu}c^{eh}_{\nu]}) \\
&\quad + \frac{4}{ig}\,\epsilon_{ha}\,\beta_{cdefgb}\,\beta_{klmnop}\,j^{cf}\,j^{dg}\,j^{kn}\,j^{lo} \\
&\qquad \times \{(\partial_{[\mu}j^{ep})(\partial_{\nu]}j^{mh}) - c^{ep}_{[\mu}\,(\partial_{\nu]}j^{mh}) + (\partial_{[\mu}j^{ep})\,c^{mh}_{\nu]} - c^{ep}_{[\mu}\,c^{mh}_{\nu]}\}\,.
\end{aligned}
\tag{4.18}
$$

Moreover, among the quantities c^{ab}_μ only the ${c_{\mu K}}^L$ (and their complex conjugate ${c_\mu}^{\dot L}{}_{\dot K} = \overline{{c_{\mu K}}^L}$) are independent. The remaining quantities, $c_{\mu K\dot L}$ and ${c_\mu}^{\dot K L}$, have to be eliminated in (4.17) due to the following identities:

$$
\begin{aligned}
c_{\mu K\dot L} &= (Q(j)^{-1})_{K\dot M}\Big\{\left({\delta^{\dot M}}_{\dot N}\,\chi^2 - {Q(j)^{\dot M}}_{\dot N}\right){c_\mu}^{\dot N}{}_{\dot L} - (\partial_\mu j_{N\dot L})\,Q(j)^{\dot M N} \\
&\quad -(\partial_\mu {j^{\dot N}}_{\dot L})\,{Q(j)^{\dot M}}_{\dot N} + (\partial_\mu {j^{\dot M}}_{\dot L})\,\chi^2\Big\}
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
{c_\mu}^{\dot K L} &= (Q(j)^{-1})^{\dot K M}\Big\{\left({\delta_M}^N\,\chi^2 - {Q(j)_M}^N\right){c_{\mu N}}^L - (\partial_\mu {j_N}^L)\,{Q(j)_M}^N \\
&\quad -(\partial_\mu j^{\dot N L})\,Q(j)_{M\dot N} + (\partial_\mu {j_M}^L)\,\chi^2\Big\}.
\end{aligned}
\tag{4.20}
$$

Proof. Using (4.2) we can rewrite (2.6) as follows:

$$
\begin{aligned}
\mathcal{F} &= \int \prod \mathrm{d}\psi \prod \mathrm{d}\overline{\psi} \prod \mathrm{d}A\,\mathrm{e}^{i\,S[A,\psi,\overline{\psi}]} \\
&\quad \times \int \prod \mathrm{d}j\,\mathrm{d}c\,\delta(j-\mathcal{J})\,\delta(c-C).
\end{aligned}
\tag{4.21}
$$

Using Proposition 2, the first Remark after Proposition 2 and Lemma 3, we get under the functional integral $\mathcal{L} = \mathcal{L}[j,c]$.

In a next step we integrate out the components in the diagonal blocks of the covector field c^{ab}_μ. We have

$$
\begin{aligned}
&\delta(c-C) \\
&\equiv \prod_{\substack{\mu=1\dots4\\ a,b=1,2,\dot 1,\dot 2}} \delta\left(c^{ab}_\mu - C^{ab}_\mu\right) \qquad (4.22)\\
&= \prod_{\substack{\mu=1\dots4\\ K,L=1,2}} \delta\left({c_{\mu K}}^L - {C_{\mu K}}^L\right)\,\delta\left({c_\mu}^{\dot K}{}_{\dot L} - {C_\mu}^{\dot K}{}_{\dot L}\right)\,\delta\left(c_{\mu K\dot L} - C_{\mu K\dot L}\right)\,\delta\left({c_\mu}^{\dot K L} - {C_\mu}^{\dot K L}\right).
\end{aligned}
$$

Using identites (2.24) and (2.25), together with (4.4), we get

$$
\prod_{\substack{\mu=1\dots4\\ K,L=1,2}} \delta\left(c_{\mu K\dot L} - C_{\mu K\dot L}\right)\,\delta\left({c_\mu}^{\dot K L} - {C_\mu}^{\dot K L}\right)
$$

$$
\begin{aligned}
= & \prod_{\substack{\mu=1\ldots4\\K,L=1,2}} \delta\left(c_{\mu K\dot{L}} - (Q(j)^{-1})_{K\dot{N}} Q(j)^{\dot{N}M} C_{\mu M\dot{L}}\right)\, \delta\left({c_\mu}^{\dot{K}L} - (Q(j)^{-1})^{\dot{K}N} Q(j)_{N\dot{M}} {C_\mu}^{\dot{M}L}\right) \\
= & \prod_{\substack{\mu=1\ldots4\\K,L=1,2}} \delta\left(c_{\mu K\dot{L}} - (Q(j)^{-1})_{K\dot{N}} Q(\mathcal{J})^{\dot{N}M} C_{\mu M\dot{L}}\right) \\
& \quad \times \delta\left({c_\mu}^{\dot{K}L} - (Q(j)^{-1})^{\dot{K}N} Q(\mathcal{J})_{N\dot{M}} {C_\mu}^{\dot{M}L}\right) \\
= & \prod_{\substack{\mu=1\ldots4\\K,L=1,2}} \delta\Big(c_{\mu K\dot{L}} - (Q(j)^{-1})_{K\dot{N}} \Big\{ \left({\delta^{\dot{N}}}_{\dot{M}}\, \mathcal{X}^2 - {Q(\mathcal{J})^{\dot{N}}}_{\dot{M}}\right) {{C_\mu}^{\dot{M}}}_{\dot{L}} \\
& \qquad - (\partial_\mu \mathcal{J}_{M\dot{L}})\, Q(\mathcal{J})^{\dot{N}M} - (\partial_\mu {\mathcal{J}^{\dot{M}}}_{\dot{L}})\, {Q(\mathcal{J})^{\dot{N}}}_{\dot{M}} + (\partial_\mu {\mathcal{J}^{\dot{N}}}_{\dot{L}})\, \mathcal{X}^2 \Big\}\Big) \\
& \quad \times \delta\Big({c_\mu}^{\dot{K}L} - (Q(j)^{-1})^{\dot{K}N} \Big\{ \left({\delta_N}^M\, \mathcal{X}^2 - {Q(\mathcal{J})_N}^M\right) {C_{\mu M}}^L \\
& \qquad - (\partial_\mu {\mathcal{J}_M}^L)\, {Q(\mathcal{J})_N}^M - (\partial_\mu \mathcal{J}^{\dot{M}L})\, Q(\mathcal{J})_{N\dot{M}} + (\partial_\mu {\mathcal{J}_N}^L)\, \mathcal{X}^2 \Big\}\Big) \\
= & \prod_{\substack{\mu=1\ldots4\\K,L=1,2}} \delta\Big(c_{\mu K\dot{L}} - (Q(j)^{-1})_{K\dot{N}} \Big\{ \left({\delta^{\dot{N}}}_{\dot{M}}\, \chi^2 - {Q(j)^{\dot{N}}}_{\dot{M}}\right) {{C_\mu}^{\dot{M}}}_{\dot{L}} \\
& \qquad - (\partial_\mu j_{M\dot{L}})\, Q(j)^{\dot{N}M} - (\partial_\mu {j^{\dot{M}}}_{\dot{L}})\, {Q(j)^{\dot{N}}}_{\dot{M}} + (\partial_\mu {j^{\dot{N}}}_{\dot{L}})\, \chi^2 \Big\}\Big) \\
& \quad \times \delta\Big({c_\mu}^{\dot{K}L} - (Q(j)^{-1})^{\dot{K}N} \Big\{ \left({\delta_N}^M\, \chi^2 - {Q(j)_N}^M\right) {C_{\mu M}}^L \\
& \qquad - (\partial_\mu {j_M}^L)\, {Q(j)_N}^M - (\partial_\mu j^{\dot{M}L})\, Q(j)_{N\dot{M}} + (\partial_\mu {j_N}^L)\, \chi^2 \Big\}\Big) \\
= & \prod_{\substack{\mu=1\ldots4\\K,L=1,2}} \delta\Big(c_{\mu K\dot{L}} - (Q(j)^{-1})_{K\dot{N}} \Big\{ \left({\delta^{\dot{N}}}_{\dot{M}}\, \chi^2 - {Q(j)^{\dot{N}}}_{\dot{M}}\right) {{c_\mu}^{\dot{M}}}_{\dot{L}} \\
& \qquad - (\partial_\mu j_{M\dot{L}})\, Q(j)^{\dot{N}M} - (\partial_\mu {j^{\dot{M}}}_{\dot{L}})\, {Q(j)^{\dot{N}}}_{\dot{M}} + (\partial_\mu {j^{\dot{N}}}_{\dot{L}})\, \chi^2 \Big\}\Big) \\
& \quad \times \delta\Big({c_\mu}^{\dot{K}L} - (Q(j)^{-1})^{\dot{K}N} \Big\{ \left({\delta_N}^M\, \chi^2 - {Q(j)_N}^M\right) {c_{\mu M}}^L \\
& \qquad - (\partial_\mu {j_M}^L)\, {Q(j)_N}^M - (\partial_\mu j^{\dot{M}L})\, Q(j)_{N\dot{M}} + (\partial_\mu {j_N}^L)\, \chi^2 \Big\}\Big).
\end{aligned}
$$

In the last step we used the first two δ-functions of (4.22) to substitute ${{C_\mu}^{\dot{M}}}_{\dot{L}}$ and ${C_{\mu M}}^L$ by its corresponding c-number quantities. $(Q(j)^{-1})_{K\dot{N}}$ and $(Q(j)^{-1})^{\dot{K}N}$ denote the inverse of the 2×2-matrices $Q(j)^{\dot{N}K}$ and $Q(j)_{N\dot{K}}$, respectively, and χ^2 is given by (4.7). Now we can perform the integration over $c_{\mu K\dot{L}}$ and ${c_\mu}^{\dot{K}L}$, which yields

$$
\mathcal{F} \;=\; \int \prod \mathrm{d}\psi \prod \mathrm{d}\overline{\psi} \prod \mathrm{d}A \int \prod \mathrm{d}j\, \delta(j - \mathcal{J})
$$

$$\times \int \prod_{\substack{\mu=1\dots4\\K,L=1,2}} \mathrm{d}c_{\mu K}{}^{L}\, \mathrm{d}c_{\mu}{}^{\dot K}{}_{\dot L}\, \delta(c_{\mu K}{}^{L} - C_{\mu K}{}^{L})\, \delta(c_{\mu}{}^{\dot K}{}_{\dot L} - C_{\mu}{}^{\dot K}{}_{\dot L})\, \mathrm{e}^{i\,S[j,c]}, \quad (4.23)$$

where $S[j,c] \equiv S[j, c_{\mu K}{}^{L}, c_{\mu}{}^{\dot K}{}_{\dot L}]$.

In a next step we integrate out the gauge potential $A_{\mu A}{}^{B}$. Observe, that $A_{\mu A}{}^{B}$ enters $\mathcal{F}$ only under the δ-distributions $\delta(c_{\mu K}{}^{L} - C_{\mu K}{}^{L})$ and $\delta(c_{\mu}{}^{\dot K}{}_{\dot L} - C_{\mu}{}^{\dot K}{}_{\dot L})$. Using (2.31), we get

$$\begin{aligned} \mathcal{F} \;&=\; \int \prod \mathrm{d}\psi \prod \mathrm{d}\overline{\psi} \prod_{\substack{\mu=1\dots4\\ \alpha=1\dots8}} \mathrm{d}A_{\mu\alpha} \int \prod \mathrm{d}j\, \delta(j-\mathcal{J}) \\ &\quad \times \int \prod_{\substack{\mu=1\dots4\\ \rho=1\dots8}} \mathrm{d}c_{\mu\rho}\, \delta(c_{\mu\rho} - \mathcal{Y}_{\rho}{}^{\alpha} A_{\mu\alpha} - f_{\mu\rho}(\psi,\overline{\psi}\,))\, \mathrm{e}^{i\,S[j,c]}, \end{aligned} \quad (4.24)$$

where

$$\mathcal{Y}_{\rho}{}^{\alpha} := 2ig\, \overline{\psi^{a}_{A}}\, g^{AB}\, \psi_{bC}\, (e_{\rho})_{a}{}^{b}\, (t^{\alpha})_{B}{}^{C} \quad (4.25)$$

is a non-singular 8×8-matrix, and

$$f_{\mu\rho}(\psi,\overline{\psi}) = \left(\overline{\psi^{a}_{A}}\, g^{AB}\, (\partial_{\mu}\psi_{bB}) + \psi_{bB}\, g^{AB}\, (\partial_{\mu}\overline{\psi^{a}_{A}}\,) \right)\, (e_{\rho})_{a}{}^{b}, \quad (4.26)$$

which is a function of ψ and $\overline{\psi}$ only. Inserting $1 \equiv \int \prod_{\substack{\alpha=1\dots8\\ \rho=1\dots8}} \mathrm{d}y_{\rho}{}^{\alpha}\, \delta(y_{\rho}{}^{\alpha} - \mathcal{Y}_{\rho}{}^{\alpha})$, where $y_{\rho}{}^{\alpha}$ is the c–number mate associated with $\mathcal{Y}_{\rho}{}^{\alpha}$, under the functional integral we have

$$\begin{aligned} \delta\left(c_{\mu\rho} - \mathcal{Y}_{\rho}{}^{\alpha} A_{\mu\alpha} - f_{\mu\rho}(\psi,\overline{\psi})\right) \;&=\; \delta\left(c_{\mu\rho} - \mathcal{Y}_{\rho}{}^{\beta} y^{\sigma}{}_{\beta} (y^{-1})^{\alpha}{}_{\sigma}\, A_{\mu\alpha} - f_{\mu\rho}(\psi,\overline{\psi}\,)\right) \\ &=\; \delta\left(c_{\mu\rho} - \mathcal{Y}_{\rho}{}^{\beta} \mathcal{Y}^{\sigma}{}_{\beta} (y^{-1})^{\alpha}{}_{\sigma}\, A_{\mu\alpha} - f_{\mu\rho}(\psi,\overline{\psi}\,)\right). \end{aligned}$$

Here $(y^{-1})^{\alpha}{}_{\rho}$ denotes the inverse of the $y^{\rho}{}_{\alpha}$. A simple calculation shows, that

$$\begin{aligned} \mathcal{Y}_{\rho}{}^{\beta}\mathcal{Y}^{\sigma}{}_{\beta} \;&=\; -4g^{2}\, \overline{\psi^{a}_{A}}\, g^{AB}\, \psi_{bC}\, (e_{\rho})_{a}{}^{b}\, (t^{\beta})_{B}{}^{C}\, \overline{\psi^{c}_{D}}\, g^{DE}\, \psi_{dF}\, (e^{\sigma})_{c}{}^{d}\, (t_{\beta})_{E}{}^{F} \\ &=\; 4g^{2} \left\{ \frac{2}{3}\, \overline{\psi^{a}_{A}} g^{AB}\, \psi_{bB}\, \overline{\psi^{c}_{D}}\, g^{DE}\, \psi_{dE} - 2\, \overline{\psi^{a}_{A}} g^{AB}\, \psi_{bE}\, \overline{\psi^{c}_{D}}\, g^{DE}\, \psi_{dB} \right\} (e_{\rho})_{a}{}^{b}\, (e^{\sigma})_{c}{}^{d} \\ &=\; 4g^{2} \left\{ \frac{2}{3}\, \mathcal{J}^{a}{}_{b} \mathcal{J}^{c}{}_{d} + 2\, \mathcal{J}^{a}{}_{d} \mathcal{J}^{c}{}_{b} \right\} (e_{\rho})_{a}{}^{b}\, (e^{\sigma})_{c}{}^{d} \\ &\equiv\; M(\mathcal{J})_{\rho}{}^{\sigma}, \end{aligned}$$

where we have used the following property of the Gell–Mann–matrices:

$$(t^{\beta})_{B}{}^{C}\, (t_{\beta})_{E}{}^{F} = -\,\frac{2}{3}\, \delta_{B}{}^{C}\, \delta_{E}{}^{F} + 2\, \delta_{B}{}^{F}\, \delta_{E}{}^{C}. \quad (4.27)$$

A long, but straightforward calculation shows that $M(\mathcal{J})_\rho{}^\sigma$ is a non-singular 8×8-matrix. Thus we obtain

$$\begin{aligned}
\delta\left(c_{\mu\rho} - \mathcal{Y}_\rho{}^\alpha A_{\mu\alpha} - f_{\mu\rho}(\psi,\overline{\psi}\,)\right) &= \delta\left(c_{\mu\rho} - M(\mathcal{J})_\rho{}^\sigma (y^{-1})^\alpha{}_\sigma\, A_{\mu\alpha} - f_{\mu\rho}(\psi,\overline{\psi}\,)\right) \\
&= \delta\left(c_{\mu\rho} - M(j)_\rho{}^\sigma (y^{-1})^\alpha{}_\sigma\, A_{\mu\alpha} - f_{\mu\rho}(\psi,\overline{\psi}\,)\right).
\end{aligned}$$

Now, performing the transformation

$$\tilde{A}_{\mu\rho} = M(j)_\rho{}^\sigma (y^{-1})^\alpha{}_\sigma\, A_{\mu\alpha}, \tag{4.28}$$

the functional integral (4.24) takes the form

$$\begin{aligned}
\mathcal{F} &= \int \prod \mathrm{d}\psi \prod \mathrm{d}\overline{\psi} \prod_{\substack{\mu=1\ldots 4\\ \rho=1\ldots 8}} \mathrm{d}\tilde{A}_{\mu\rho} \int \prod \mathrm{d}j\, \delta(j-\mathcal{J}) \int \prod_{\substack{\alpha=1\ldots 8\\ \rho=1\ldots 8}} \mathrm{d}y_\rho{}^\alpha\, \delta(y_\rho{}^\alpha - \mathcal{Y}_\rho{}^\alpha) \\
&\quad\times \int \prod_{\substack{\mu=1\ldots 4\\ \rho=1\ldots 8}} \mathrm{d}c_{\mu\rho}\, \delta\Big(c_{\mu\rho} - \tilde{A}_{\mu\rho} - f_{\mu\rho}(\psi,\overline{\psi}\,)\Big) \left(\det\left[M(j)_\rho{}^\sigma (y^{-1})^\alpha{}_\sigma\right]\right)^{-1} \mathrm{e}^{i\,S[j,c]} \\
&= \int \prod \mathrm{d}\psi \prod \mathrm{d}\overline{\psi} \prod_{\substack{\mu=1\ldots 4\\ \rho=1\ldots 8}} \mathrm{d}\tilde{A}_{\mu\rho} \int \prod \mathrm{d}j\, \delta(j-\mathcal{J}) \int \prod_{\substack{\alpha=1\ldots 8\\ \rho=1\ldots 8}} \mathrm{d}y_\rho{}^\alpha\, \delta(y_\rho{}^\alpha - \mathcal{Y}_\rho{}^\alpha) \\
&\quad\times \int \prod_{\substack{\mu=1\ldots 4\\ \rho=1\ldots 8}} \mathrm{d}c_{\mu\rho}\, \delta\Big(c_{\mu\rho} - \tilde{A}_{\mu\rho} - f_{\mu\rho}(\psi,\overline{\psi}\,)\Big) \frac{1}{\det[M(j)_\rho{}^\sigma]} \det\left[y^\sigma{}_\alpha\right] \mathrm{e}^{i\,S[j,c]}.
\end{aligned}$$

Now we can trivially integrate out $\tilde{A}_{\mu\rho}$ and the auxiliary field $y_\rho{}^\alpha$. We get

$$\mathcal{F} = \int \prod \mathrm{d}\psi \prod \mathrm{d}\overline{\psi} \int \prod \mathrm{d}j\, \delta(j-\mathcal{J}) \int \prod_{\substack{\mu=1\ldots 4\\ \rho=1\ldots 8}} \mathrm{d}c_{\mu\rho} \frac{1}{\det[M(j)_\rho{}^\sigma]} \det\left[\mathcal{Y}^\sigma{}_\alpha\right] \mathrm{e}^{i\,S[j,c]}.$$

It remains to calculate $\det[\mathcal{Y}^\sigma{}_\alpha]$. A very long but straightforward calculation shows, that $(\mathcal{J}^2)^2 \det[\mathcal{Y}^\sigma{}_\alpha]$ is a nonvanishing element of maximal rank. Thus – due to Lemma 1 – there exists a nonzero real number a such that

$$(\mathcal{J}^2)^2 \det\left[\mathcal{Y}^\sigma{}_\alpha\right] = a\,(\mathcal{X}^2)^4. \tag{4.29}$$

Inserting – due to (4.4) – an additional factor $\int \prod \mathrm{d}\tilde{j}\,\delta(j-\tilde{j})$ under the functional integral we can write

$$\det\left[\mathcal{Y}^\sigma{}_\alpha\right] = \frac{(j^2)^2}{(\tilde{j}^2)^2} \det\left[\mathcal{Y}^\sigma{}_\alpha\right] = \frac{(\mathcal{J}^2)^2}{(\tilde{j}^2)^2} \det\left[\mathcal{Y}^\sigma{}_\alpha\right] = a\frac{(\mathcal{X}^2)^4}{(\tilde{j}^2)^2}. \tag{4.30}$$

Now we can integrate out the auxiliary quantity $\tilde{j}$ and obtain

$$\mathcal{F} = \int \prod \mathrm{d}\psi \prod \mathrm{d}\overline{\psi} \int \prod \mathrm{d}j\, \mathrm{d}c\, \delta(j-\mathcal{J}) \frac{(\chi^2)^4}{\det[M(j)_\rho{}^\sigma]\,(j^2)^2} \mathrm{e}^{i\,S[j,c]},$$

where χ^2 and j^2 are given by (4.7) and (4.8), respectively. The number a has to be absorbed in the (global) normalization factor, which – any way – is omitted here.

The remaining gauge dependent fields ψ and $\overline{\psi}$ occur only in the δ-distributions. To integrate them out we use the integral representation (4.1), i.e. we insert

$$\begin{aligned}
\delta(j-\mathcal{J}) &= \int \prod \mathrm{d}\lambda \exp\left[2\pi i\, \lambda_{ab}\,(j^{ab}-J^{ab})\right] \\
&= \int \prod \mathrm{d}\lambda\, \mathrm{e}^{2\pi i\, \lambda_{ab} j^{ab}} \sum_{n=0}^{12} \frac{(2\pi i)^n}{n!} (-\lambda_{ab}\mathcal{J}^{ab})^n.
\end{aligned}$$

Observe that nonvanishing contributions will come from terms which are of order 12 both in ψ and $\overline{\psi}$. We get

$$\begin{aligned}
\mathcal{F} &= \int \prod \mathrm{d}\psi \prod \mathrm{d}\overline{\psi} \int \prod \mathrm{d}j\, \mathrm{d}c\, \frac{(\chi^2)^4}{\det[M(j)_\rho{}^\sigma]\,(j^2)^2}\, \mathrm{e}^{i\,S[j,c]} \\
&\quad \times \int \prod \mathrm{d}\lambda\, \mathrm{e}^{2\pi i\, \lambda j}\, (-1)^{12}\, \frac{(2\pi i)^{12}}{12!} \left(\lambda_{ab}\, J^{ab}\right)^{12} \\
&= \int \prod \mathrm{d}j\, \mathrm{d}c\, \frac{(2\pi)^{12}}{12!} \frac{(\chi^2)^4}{\det[M(j)_\rho{}^\sigma]\,(j^2)^2}\, \mathrm{e}^{i\,S[j,c]} \int \prod \mathrm{d}\lambda\, \mathrm{e}^{2\pi i\, \lambda j} \prod_{i=1}^{12} \lambda_{a_i b_i} \\
&\quad \times \int \prod \mathrm{d}\psi \prod \mathrm{d}\overline{\psi} \prod_{i=1}^{12} J^{a_i b_i}.
\end{aligned}$$

Now we can integrate out ψ and $\overline{\psi}$ using equation (B. 9). Replacing the factors λ_{ab} by corresponding derivatives $\frac{\partial}{\partial j^{ab}}$ yields

$$\mathcal{F} = \int \prod \mathrm{d}j\, \mathrm{d}c\, K[j]\, \mathrm{e}^{i\,S[j,c]} \int \prod \mathrm{d}\lambda\, \mathrm{e}^{2\pi i\, \lambda j},$$

where

$$\begin{aligned}
K[j] &= \frac{6^4}{12!} \frac{(\chi^2)^4}{\det[M(j)_\rho{}^\sigma]\,(j^2)^2}\, \epsilon_{c_1 c_4 c_7 c_{10}}\, \epsilon_{c_2 c_5 c_8 c_{11}}\, \epsilon_{c_3 c_6 c_9 c_{12}}\, \epsilon^{b_1 b_4 b_7 b_{10}}\, \epsilon^{b_2 b_5 b_8 b_{11}}\, \epsilon^{b_3 b_6 b_9 b_{12}} \\
&\quad \left\{ \times \prod_{r=1}^{12} \beta^{a_r c_r} \frac{\partial}{\partial j^{a_r b_r}} \right\} \int \prod \mathrm{d}\lambda\, \mathrm{e}^{2\pi i\, \lambda j}.
\end{aligned}$$

With

$$\int \prod \mathrm{d}\lambda\, \mathrm{e}^{2\pi i\, \lambda j} = \delta(j)$$

we can perform the integration over λ, which finally proves the Proposition. □

We remark, that until now we considered only the “bare” functional integral (2.6). To calculate the vacuum expectation value for some observable, one has to insert this

observable, i.e. a gauge invariant function $\mathcal{O}[A,\psi,\overline{\psi}]$, under the above integral. With the tools given in the last sections we see, that this function can be written down in the form $\mathcal{O} = \mathcal{O}[C_\mu^{ab}, \mathcal{J}^{ab}]$. Our method of changing variables used in the proof of Proposition 2 works in the same way for vacuum expectation values of observables of this type, yielding an additional factor $\mathcal{O}[c_\mu^{ab}, j^{ab}]$. This shows that, indeed, we are dealing with a reduced theory: vacuum expectation values of baryons (trilinear combinations of quarks) can – in general – not be calculated using the above functional integral. Only certain combinations, namely – roughly speaking – such, which are expressable in terms of the j-field, may be treated this way. This is due to the fact that there exist certain identities relating bilinear combinations of quarks and antiquarks at one hand and trilinear combinations of quarks and their complex conjugates on the other hand, see formula (D.7) of [12].

A Spinorial structures

Since we are going to work with multilinear (and not only bilinear) expressions in spinor fields, the ordinary matrix notation is not sufficient for our purposes. Therefore, we will have to use a consequent tensorial calculus in bispinor space. For those, who are not familiar with this language we give a short review of its basic notions. A bispinor will be represented by:

$$\psi^a = \begin{pmatrix} \phi^K \\ \varphi_{\dot K} \end{pmatrix} \equiv \begin{pmatrix} \phi^1 \\ \phi^2 \\ \varphi_{\dot 1} \\ \varphi_{\dot 2} \end{pmatrix}, \tag{A. 1}$$

where ϕ^K is a Weyl spinor belonging to the spinor space $S \cong \mathbb{C}^2$, carrying the fundamental representation of $SL(2,\mathbb{C})$. Besides S we have to consider the spaces S^*, $\bar S$ and $\bar S^*$, where $*$ denotes the algebraic dual and bar denotes the complex conjugate. All these spaces are isomorphic to S, but carry different representations of $SL(2,\mathbb{C})$. In S^* acts the dual (equivalent to the fundamental) representation and in $\bar S$ acts the conjugate (not equivalent) representation of $SL(2,\mathbb{C})$. The space S is equipped with an $SL(2,\mathbb{C})$-invariant, skew-symmetric bilinear form ϵ_{KL}. Since it is non-degenerate, it gives an isomorphism between S and S^*:

$$S \ni \left(\phi^K\right) \mapsto (\phi_L) = \left(\phi^K \epsilon_{KL}\right) \in S^*. \tag{A. 2}$$

There is also a canonical anti-isomorphism between S and $\bar S$ given by complex conjugation:

$$S \ni \left(\phi^K\right) \mapsto \left(\bar\phi^{\dot K}\right) \equiv \left(\overline{\phi^K}\right) \in \bar S. \tag{A. 3}$$

Finally, the conjugate bilinear form $\epsilon_{\dot K \dot L}$ gives an isomorphism between $\bar S$ and $\bar S^*$:

$$\bar S \ni \left(\varphi^{\dot K}\right) \mapsto (\varphi_{\dot L}) = \left(\varphi^{\dot K} \epsilon_{\dot K \dot L}\right) \in \bar S^*. \tag{A. 4}$$

To summarize, we have the following commuting diagram:

$$\begin{array}{ccc} S & \longrightarrow & \bar{S} \\ \downarrow & & \downarrow \\ S^* & \longrightarrow & \bar{S}^* \end{array} . \tag{A. 5}$$

Formula (A. 1) means that a bispinor is an element of $\mathcal{S} := S \times \bar{S}^*$, carrying the product of the fundamental and the dual to the conjugate representation of $SL(2,\mathbb{C})$. We also consider the complex conjugate bispinor

$$\bar{\psi}^a = \begin{pmatrix} \bar{\phi}^{\dot{L}} \\ \bar{\varphi}_L \end{pmatrix} \equiv \begin{pmatrix} \bar{\phi}^{\dot{1}} \\ \bar{\phi}^{\dot{2}} \\ \bar{\varphi}_1 \\ \bar{\varphi}_2 \end{pmatrix} , \tag{A. 6}$$

belonging to the conjugate space $\bar{\mathcal{S}} = \bar{S} \times S^*$. The tensor product of ϵ_{KL} and $-\epsilon^{\dot{K}\dot{L}}$ defines a skew symmetric bilinear form ϵ_{ab} on $\mathcal{S}$, which in turn gives an isomorphism between $\mathcal{S}$ and $\mathcal{S}^*$:

$$\mathcal{S} \ni (\psi^a) \mapsto (\psi_b) = (\psi^a \, \epsilon_{ab}) \in \mathcal{S}^* , \tag{A. 7}$$

with

$$\epsilon_{ab} = \left(\begin{array}{c|c} \epsilon_{KL} & 0 \\ \hline 0 & -\epsilon^{\dot{K}\dot{L}} \end{array} \right) . \tag{A. 8}$$

We choose the minus sign in the lower block, because lowering the bispinor index $\uparrow a = (\uparrow K, \downarrow \dot{K})$ according to (A. 7) means lowering K and rising $\dot{K}$. But rising an index needs $-\epsilon$, because we have

$$\epsilon_{\dot{K}\dot{L}} \, \epsilon^{\dot{L}\dot{M}} = -\delta_{\dot{K}}{}^{\dot{M}} . \tag{A. 9}$$

The natural algebraic duality between $\mathcal{S}$ and $\bar{\mathcal{S}} = \bar{S} \times S^* \cong S^* \times \bar{S}$ defines a Hermitian bilinear form β_{ab} given by

$$\bar{\psi}^a_{(1)} \, \beta_{ab} \psi^b_{(2)} = \bar{\phi}^{\dot{K}}_{(1)} \, \varphi_{(2)\dot{K}} + \bar{\varphi}_{(1)K} \, \phi^K_{(2)} . \tag{A. 10}$$

Thus,

$$\beta_{ab} = \left(\begin{array}{c|c} 0 & \delta_{\dot{K}}{}^{\dot{L}} \\ \hline \delta^K{}_L & 0 \end{array} \right) . \tag{A. 11}$$

(We stress that a and b are different indices: $\downarrow a = (\downarrow \dot{K}, \uparrow K)$ is a conjugate index corresponding to $\bar{\mathcal{S}}$ and $\downarrow b = (\downarrow L, \uparrow \dot{L})$ is an index from $\mathcal{S}$.) The relations between spin tensors and space time objects are given by the Dirac γ-matrices, which we use in the chiral representation

$$\gamma^\mu = \left(\begin{array}{c|c} 0 & \tilde{\sigma}^\mu \\ \hline \sigma^\mu & 0 \end{array} \right) , \tag{A. 12}$$

with

$$\tilde{\sigma}^\mu = -\epsilon \, \bar{\sigma}^\mu \, \epsilon = \left(\mathbf{1}, -\sigma^k \right) . \tag{A. 13}$$

In index notation we have

$$\sigma^\mu = ({\sigma^\mu}_{\dot{K}L}) , \tag{A. 14}$$

and

$$\tilde{\sigma}^\mu = \left(\tilde{\sigma}^{\mu M \dot{N}} \right) , \tag{A. 15}$$

with

$$\tilde{\sigma}^{\mu M \dot{N}} = -\epsilon^{MK} \, \overline{{\sigma^\mu}_{\dot{K}L}} \, \epsilon^{\dot{L}\dot{N}} = \epsilon^{MK} \, \epsilon^{\dot{N}\dot{L}} \, \bar{\sigma}^\mu_{K\dot{L}} = \epsilon^{MK} \, \epsilon^{\dot{N}\dot{L}} \, {\sigma^\mu}_{\dot{L}K} = \sigma^{\mu \dot{N} M} . \tag{A. 16}$$

Finally, we get

$$(\gamma^\mu)^a{}_b = \left(\begin{array}{c|c} 0 & \sigma^{\mu \dot{L} K} \\ \hline {\sigma^\mu}_{\dot{K}L} & 0 \end{array} \right) , \tag{A. 17}$$

where $\uparrow a = (\uparrow K, \downarrow \dot{K})$ and $\downarrow b = (\downarrow L, \uparrow \dot{L})$. After pulling down the first index by the help of ϵ_{ac}, where $\downarrow c = (\downarrow M, \uparrow \dot{M})$, we get

$$(\gamma^\mu)_{cb} = (\gamma^\mu)^a{}_b \, \epsilon_{ac} = \left(\begin{array}{c|c} 0 & {\sigma^{\mu \dot{L}}}_M \\ \hline {\sigma^{\mu \dot{M}}}_L & 0 \end{array} \right) , \tag{A. 18}$$

which is a symmetric bilinear form, because

$$\psi^c_{(1)} \, {\gamma^\mu}_{cb} \, \psi^b_{(2)} = \phi^M_{(1)} \, {\sigma^{\mu \dot{L}}}_M \, \varphi_{(2)\dot{L}} + \varphi_{(1)\dot{M}} \, {\sigma^{\mu \dot{M}}}_L \, \phi^L_{(2)} . \tag{A. 19}$$

The complex conjugate quantity is given by

$$\overline{(\gamma^\mu)_{cb}} = \left(\begin{array}{c|c} 0 & \sigma^\mu{}_{\dot M}{}^L \\ \hline \sigma^\mu{}_{\dot L}{}^M & 0 \end{array}\right). \tag{A. 20}$$

We also use the following spin tensor

$$\beta_{abcd} := \frac{1}{2}\overline{(\gamma^\mu)_{ab}}\,(\gamma_\mu)_{cd}\,. \tag{A. 21}$$

Obviously, this tensor is symmetric in the first and in the second pair of indices seperately. We see from (A. 19) and (A. 20) that it vanishes, whenever a and b or c and d are of the same type (both dotted or both undotted). Thus, the only nonvanishing components are

$$\beta_{\dot K}{}^{L\dot M}{}_N = \beta_{\dot K}{}^L{}_N{}^{\dot M} = \beta^L{}_{\dot K}{}^{\dot M}{}_N = \beta^L{}_{\dot K N}{}^{\dot M} = -\delta^{\dot M}{}_{\dot K}\,\delta^L{}_N\,. \tag{A. 22}$$

Finally, observe that second rank spin tensors have a natural block structure. In particular, for the invariants considered in this paper we get:

$$\mathcal{J}^{ab} = \left(\begin{array}{c|c} \mathcal{J}^{\dot K L} & \mathcal{J}^{\dot K}{}_{\dot L} \\ \hline \mathcal{J}_K{}^L & \mathcal{J}_{K\dot L} \end{array}\right), \tag{A. 23}$$

where $\mathcal{J}^{\dot K L}$, $\mathcal{J}_{K\dot L}$, $\mathcal{J}^{\dot K}{}_{\dot L}$ and $\mathcal{J}_K{}^L$ are Hermitean 2×2-matrices. Analogously,

$$C^{ab}_\mu = \left(\begin{array}{c|c} C^{\dot K L}_\mu & C_\mu{}^{\dot K}{}_{\dot L} \\ \hline C_{\mu K}{}^L & C_{\mu K\dot L} \end{array}\right). \tag{A. 24}$$

B Calculation of $(\mathcal{X}^2)^4$

Obviously $(\mathcal{X}^2)^4$ is an element of maximal rank in the Grassmann-algebra, that is

$$(\mathcal{X}^2)^4 = c\prod\psi\prod\bar\psi. \tag{B. 1}$$

To prove that it is nonzero it remains to calculate the number $c \in \mathbb{C}$ and to show, that $c \neq 0$. From (B. 1) it follows by integration that

$$c = \int (\mathcal{X}^2)^4 \prod \mathrm{d}\psi \prod \mathrm{d}\bar{\psi}. \tag{B. 2}$$

Using the definition of $\mathcal{X}^2$ we have

$$(\mathcal{X}^2)^4 = \prod_{r=1}^{4} 4 \left(\mathcal{J}^{a_r d_r} \mathcal{J}^{b_r e_r} \mathcal{J}^{c_r f_r} + 2\, \mathcal{J}^{a_r f_r} \mathcal{J}^{b_r d_r} \mathcal{J}^{c_r e_r}\right) \beta_{b_r c_r e_r f_r} \beta_{a_r d_r}.$$

Inserting this into (B. 2) we can decompose c into a sum

$$c = c_1 + c_2 + c_3 + c_4 + c_5, \tag{B. 3}$$

with

$$\begin{aligned}
c_1 &= 4^4 \int \prod_{r=1}^{4} \mathcal{J}^{a_r d_r} \mathcal{J}^{b_r e_r} \mathcal{J}^{c_r f_r} \beta_{b_r c_r e_r f_r}\, \beta_{a_r d_r} \prod \mathrm{d}\psi \prod \mathrm{d}\bar{\psi} \\
&= \frac{1}{(4!)^3}\, 6^4\, (4!)^3\, 4^4 \prod_{r=1}^{4} (\beta_{b_r c_r e_r f_r}\, \beta_{a_r d_r}\, \beta^{a_r i_r}\, \beta^{b_r j_r}\, \beta^{c_r k_r}) \\
&\quad \times \epsilon_{i_1 i_4 j_3 k_2}\, \epsilon_{i_2 j_1 j_4 k_3}\, \epsilon_{i_3 j_2 k_1 k_4}\, \epsilon^{d_1 d_4 e_3 f_2}\, \epsilon^{d_2 e_1 e_4 f_3}\, \epsilon^{d_3 e_2 f_1 f_4} \\
&= 6^4\, 4^4\, U^{i_1 j_1 k_1}_{e_1 f_1 d_1}\, U^{i_2 j_2 k_2}_{e_2 f_2 d_2}\, U^{i_3 j_3 k_3}_{e_3 f_3 d_3}\, U^{i_4 j_4 k_4}_{e_4 f_4 d_4} \\
&\quad \times \epsilon_{i_1 i_4 j_3 k_2}\, \epsilon_{i_2 j_1 j_4 k_3}\, \epsilon_{i_3 j_2 k_1 k_4}\, \epsilon^{d_1 d_4 e_3 f_2}\, \epsilon^{d_2 e_1 e_4 f_3}\, \epsilon^{d_3 e_2 f_1 f_4},
\end{aligned} \tag{B. 4}$$

$$\begin{aligned}
c_2 &= 2^4\, 4^4 \int \prod_{r=1}^{4} \mathcal{J}^{a_r f_r} \mathcal{J}^{b_r d_r} \mathcal{J}^{c_r e_r} \beta_{b_r c_r e_r f_r}\, \beta_{a_r d_r} \prod \mathrm{d}\psi \prod \mathrm{d}\bar{\psi} \\
&= 6^4\, 2^4\, 4^4\, U^{i_1 j_1 k_1}_{e_1 f_1 d_1}\, U^{i_2 j_2 k_2}_{e_2 f_2 d_2}\, U^{i_3 j_3 k_3}_{e_3 f_3 d_3}\, U^{i_4 j_4 k_4}_{e_4 f_4 d_4} \\
&\quad \times \epsilon_{i_1 i_4 j_3 k_2}\, \epsilon_{i_2 j_1 j_4 k_3}\, \epsilon_{i_3 j_2 k_1 k_4}\, \epsilon^{d_1 d_4 e_3 f_2}\, \epsilon^{d_2 e_1 e_4 f_3}\, \epsilon^{d_3 e_2 f_1 f_4}, \\
&= 2^4\, c_1,
\end{aligned} \tag{B. 5}$$

$$\begin{aligned}
c_3 &= 2^3\, 4^4 \int \prod_{r=1}^{3} \mathcal{J}^{a_r d_r} \mathcal{J}^{b_r e_r} \mathcal{J}^{c_r f_r} \beta_{b_r c_r e_r f_r}\, \beta_{a_r d_r} \\
&\quad \times \mathcal{J}^{a_4 f_4} \mathcal{J}^{b_4 d_4} \mathcal{J}^{c_4 e_4}\, \beta_{b_4 c_4 e_4 f_4}\, \beta_{a_4 d_4} \prod \mathrm{d}\psi \prod \mathrm{d}\bar{\psi} \\
&= 6^4\, 2^3\, 4^4\, U^{i_1 j_1 k_1}_{e_1 f_1 d_1}\, U^{i_2 j_2 k_2}_{e_2 f_2 d_2}\, U^{i_3 j_3 k_3}_{e_3 f_3 d_3}\, U^{i_4 j_4 k_4}_{e_4 f_4 d_4} \\
&\quad \times \epsilon_{i_1 i_4 j_3 k_2}\, \epsilon_{i_2 j_1 j_4 k_3}\, \epsilon_{i_3 j_2 k_1 k_4}\, \epsilon^{d_1 f_4 e_3 f_2}\, \epsilon^{d_2 e_1 d_4 f_3}\, \epsilon^{d_3 e_2 f_1 e_4},
\end{aligned} \tag{B. 6}$$

$$c_4 = 2^2\, 4^4\, 6 \int \prod_{r=1}^{2} \mathcal{J}^{a_r d_r} \mathcal{J}^{b_r e_r} \mathcal{J}^{c_r f_r} \beta_{b_r c_r e_r f_r}\, \beta_{a_r d_r}$$

$$
\begin{aligned}
& \times \int \prod_{s=1}^{2} \mathcal{J}^{a_s f_s} \mathcal{J}^{b_s d_s} \mathcal{J}^{c_s e_s} \beta_{b_s c_s e_s f_s}\, \beta_{a_s d_s} \prod \mathrm{d}\psi \prod \mathrm{d}\bar{\psi} \\
= \; & 6^4\, 2^2\, 4^4\, 6\, U^{i_1 j_1 k_1}_{e_1 f_1 d_1}\, U^{i_2 j_2 k_2}_{e_2 f_2 d_2}\, U^{i_3 j_3 k_3}_{e_3 f_3 d_3}\, U^{i_4 j_4 k_4}_{e_4 f_4 d_4} \\
& \times \epsilon_{i_1 i_4 j_3 k_2}\, \epsilon_{i_2 j_1 j_4 k_3}\, \epsilon_{i_3 j_2 k_1 k_4}\, \epsilon^{d_1 f_4 d_3 f_2}\, \epsilon^{d_2 e_1 d_4 e_3}\, \epsilon^{f_3 e_2 f_1 e_4}, \qquad \text{(B. 7)} \\
c_5 = \; & 2^5\, 4^4 \int \prod_{r=1}^{3} \mathcal{J}^{a_r f_r} \mathcal{J}^{b_r d_r} \mathcal{J}^{c_r e_r} \beta_{b_r c_r e_r f_r}\, \beta_{a_r d_r} \\
& \times \mathcal{J}^{a_4 d_4} \mathcal{J}^{b_4 e_4} \mathcal{J}^{c_4 f_4} \beta_{b_4 c_4 e_4 f_4}\, \beta_{a_4 d_4} \prod \mathrm{d}\psi \prod \mathrm{d}\bar{\psi} \\
= \; & 6^4\, 2^5\, 4^4\, U^{i_1 j_1 k_1}_{e_1 f_1 d_1}\, U^{i_2 j_2 k_2}_{e_2 f_2 d_2}\, U^{i_3 j_3 k_3}_{e_3 f_3 d_3}\, U^{i_4 j_4 k_4}_{e_4 f_4 d_4} \\
& \times \epsilon_{i_1 i_4 j_3 k_2}\, \epsilon_{i_2 j_1 j_4 k_3}\, \epsilon_{i_3 j_2 k_1 k_4}\, \epsilon^{f_1 d_4 d_3 e_2}\, \epsilon^{f_2 d_1 e_4 e_3}\, \epsilon^{f_3 d_2 e_1 f_4}, \qquad \text{(B. 8)}
\end{aligned}
$$

where

$$
U^{ijk}_{efd} := \beta_{bcef}\, \beta_{ad}\, \beta^{ai}\, \beta^{bj}\, \beta^{ck}.
$$

Here we made use of the following equation

$$
\begin{aligned}
\int \prod \mathrm{d}\psi \prod \mathrm{d}\overline{\psi} \prod_{r=1}^{12} \mathcal{J}^{c_r d_r} \; = \; & \frac{1}{(4!)^3} \prod_{r=1}^{12} \beta^{c_r l_r} \sum_{\sigma} \epsilon_{l_{\sigma_1} l_{\sigma_2} l_{\sigma_3} l_{\sigma_4}}\, \epsilon_{l_{\sigma_5} \dots l_{\sigma_8}}\, \epsilon_{l_{\sigma_9} \dots l_{\sigma_{12}}} \\
& \times\, \epsilon^{d_{\sigma_1} d_{\sigma_2} d_{\sigma_3} d_{\sigma_4}}\, \epsilon^{d_{\sigma_5} \dots d_{\sigma_8}}\, \epsilon^{d_{\sigma_9} \dots d_{\sigma_{12}}}, \qquad \text{(B. 9)}
\end{aligned}
$$

where the sum is taken over all permutations. To prove this formula we observe, that both sides do not vanish, iff every spinor index occurs exactly three times in the multi-indices (c_r) and (d_r). Furthermore, both sides are symmetric with respect to every simultaneous transposition of two pairs of indices, e.g. (c_r, d_r) and (c_s, d_s). Therefore, the indices c_r may be ordered on both sides. After having done this, the above formula can be checked by inspection.

To prove (B. 4) – (B. 8) we note that only such terms give a nonvanishing contribution, for which all indices within every ϵ-tensor are different. The number of such permutations is $6^4\,(4!)^3$ and it is easy to see that all of them give the same contribution. Therefore, we can replace the sum over all permutations by a concrete representation multiplied by the number $6^4\,(4!)^3$.

The next step is to perform the sum over all indices in (B. 4), (B. 5), (B. 6), (B. 7) and (B. 8). A lengthy but simple tensorial calculation gives

$$
\begin{aligned}
c_1 &= 4^4\, 6^4\, 2^7\, 3^2 = 2^{19}\, 3^6, \\
c_2 &= 4^4\, 6^4\, 2^4\, 2^7\, 3^2 = 2^{23}\, 3^6, \\
c_3 &= 4^4\, 6^4\, 2^3\, 2^4\, 3^3 = 2^{19}\, 3^7, \\
c_4 &= 4^4\, 6^4\, 2^3\, 2^3\, 3^3\, 5 = 2^{18}\, 3^7\, 5, \\
c_5 &= 4^4\, 6^4\, 2^5\, 2^4\, 3^3 = 2^{21}\, 3^7.
\end{aligned}
$$

Taking the sum we get

$$c = 2^{18}\, 3^6\, 79.$$

$\square$

References

[1] J.Kijowski and G.Rudolph, Lett. Math. Phys. **29**, 103 (1993)

[2] J.Kijowski, G.Rudolph and M.Rudolph, Lett. Math. Phys. **33**, 139 (1995)

[3] L.D.Faddeev and V.N.Popov, Phys. Lett. **B 25**, 30 (1967)

[4] J.Kijowski, G.Rudolph and A.Thielmann, *"The algebra of observables and charge superselection sectors of QED on the lattice"* (in preparation)

[5] V.N.Gribov, Nucl. Phys. **B139**, 1 (1978)

[6] I.M.Singer, Comm. Math. Phys. **60**, 7 (1978)

[7] P.K.Mitter and C.M.Viallet, Comm. Math. Phys. **79**, 457 (1981) ,
O.Babelon and C.M.Viallet, Phys. Lett. **85B**, 246 (1979) ,
W.Kondracki and J.Rogulski, prepr. 62/83/160, Inst. of Math. PAN, Warsaw (1983) ,
P.K.Mitter, Cargese Lectures 1979, in G.'t Hooft et.al. (eds.), *Recent Development in Gauge Theories*, Plenum Press, New York 1980

[8] D.Zwanziger, Nucl. Phys. **B209**, 336 (1982), **B323**, 513 (1989), **B345**, 461 (1990)

[9] J.Kijowski and G.Rudolph, Nucl. Phys. **B325**, 211 (1989)

[10] G.Rudolph, Lett. Math. Phys. **16**, 27 (1988)

[11] J.Kijowski and G.Rudolph, Phys. Rev. **D31**, 856 (1985) ,
G.Rudolph, Annalen der Physik, 7.Folge, Bd. 47, 2/3, 211 (1990)

[12] J.Kijowski and G.Rudolph, *"One-Flavour Chromodynamics in Terms of Gauge Invariant Quantities"*, Leipzig University preprint Nr. 1/93 (1993)

[13] F.A.Berezin, *Metod vtoričnovo kvantovania*, Nauka, Moskva 1986

[14] F.A.Berezin, *The Method of Second Quantization*, Academic Press, New York and London 1966

[15] F.A.Berezin, *Introduction to Superanalysis*, D. Reidel Publ. Comp., Dordrecht 1987

[16] D.Leites (ed), Seminar on Supermanifolds, No. 30, 31, Reports of Stockholm University, 1988-13, 1988-14

[17] F. Treves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, Pure and Applied Math. 25, New York 1967